LECTURES 1 - 10

INTRODUCTION TO CLASSICAL MECHANICS

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MT 2016

OUTLINE : INTRODUCTION TO MECHANICS LECTURES 1-10

1.1 Outline of lectures

Two groups of lectures

- 10 in MT - mostly 1D & 2D linear motion.
- 19 in HT - 3D full vector treatment of Newtonian mechanics, rotational dynamics, orbits, introduction to Lagrangian dynamics

Info on the course is on the web:
http://www.physics.ox.ac.uk/users/harnew/lectures/

- Synopsis and suggested reading list
- Problem sets
- Lecture slides

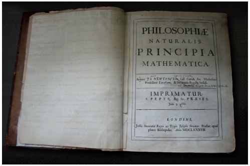
1.2 Book list

- Introduction to Classical Mechanics A P French & M G Ebison (Chapman & Hall)
- Introduction to Classical Mechanics D. Morin (CUP) (good for Lagrangian Dynamics and many examples).
- Classical Mechanics : a Modern Introduction, M W McCall (Wiley 2001)
- Mechanics Berkeley Physics Course Vol I C Kittel et al. (McGraw Hill)
- Fundamentals of Physics Halliday, Resnick & Walker (Wiley)
- Analytical Mechanics 6th ed, Fowles & Cassidy (Harcourt)
- Physics for Scientists & Engineers, (Chapters on Mechanics) P.A Tipler & G. Mosca (W H Freeman)
- Classical Mechanics T W B Kibble & F H H Berkshire (Imperial College Press)

1.3 What is Classical Mechanics?

Classical mechanics is the study of the motion of bodies in accordance with the general principles first enunciated by Sir Isaac Newton in his Philosophiae Naturalis Principia Mathematica (1687). Classical mechanics is the foundation upon which all other branches of Physics are built. It has many important applications in many areas of science:

- Astronomy (motion of stars and planets)
- Molecular and nuclear physics (collisions of atomic and subatomic particles)
- Geology (e.g., the propagation of seismic waves)
- Engineering (eg structures of bridges and buildings)



Classical Mechanics covers:

- The case in which bodies remain at rest
- Translational motion– by which a body shifts from one point in space to another
- Oscillatory motion– e.g., the motion of a pendulum or spring
- Circular motion–motion by which a body executes a circular orbit about another fixed body [e.g., the (approximate) motion of the earth about the sun]
- More general rotational motion–orbits of planets or bodies that are spinning
- Particle collisions (elastic and inelastic)

Forces in mechanics

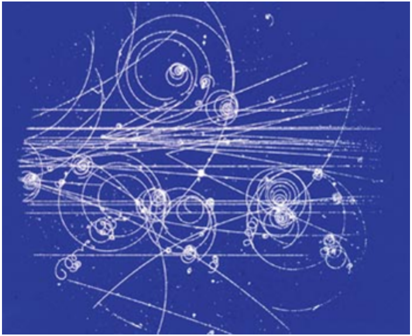
Relative magnitude of forces:

- Strong force - nuclear : ~ 1
- Electromagnetism - charged particles : $\frac{1}{137}$
- Weak force - β decay : $\sim 10^{-5}$
- Gravitational - important for masses, relative strength : $\sim 10^{-39}$

Not too fast!

Classical Mechanics valid on scales which are:

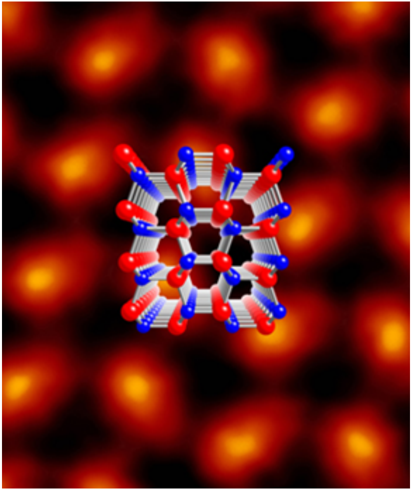
- Not too fast
- eg. high energy particle tracks from CERN
- $v << c$ [speed of light in vacuo]
- If too fast, time is no longer absolute - need special relativity.



Not too small!

Classical Mechanics valid on scales which are:

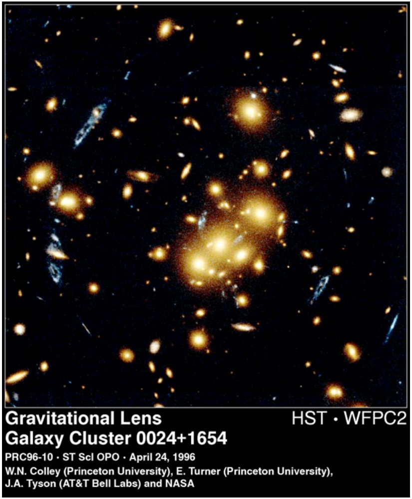
- Not too small!
- Images of atom planes in a lattice by scanning tunneling electron microscope
- Particles actually have wave-like properties : $\lambda = \frac{h}{p} \quad (h = 6.6 \times 10^{-34}\ \text{Js})$
- Hence for scales $>> \lambda$, wave properties can be ignored



Not too large!

Classical Mechanics valid on scales which are:

- Not too large!
- Gravitational lens produced by a cluster of galaxies
- Space is "flat" in classical mechanics - curvature of space is ignored
- Also in Newtonian mechanics, time is absolute



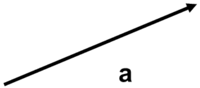
1.4 Vectors in mechanics

The use of vectors is essential in the formalization of classical mechanics.

- A scalar is characterised by magnitude only: energy, temperature.
- A vector is a quantity characterised by magnitude and direction: eg. Force, momentum, velocity.

Notation:

- Vector: **a** (bold); in components $\underline{\mathbf{a}} = (a_x, a_y, a_z)$
- Magnitude of $\underline{\mathbf{a}}$ is $|\underline{\mathbf{a}}|$ or simply a.



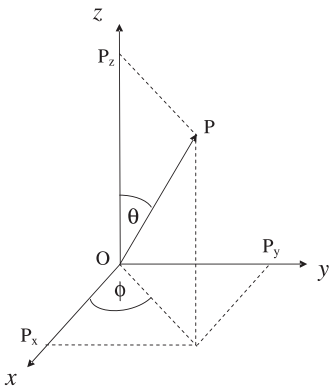
- Two vectors are equal if they have the same magnitude and direction (i.e. parallel)
  $\underline{\mathbf{a}} = \underline{\mathbf{b}}$ gives $a_x = b_x, \ a_y = b_y, \ a_z = b_z$

1.4.1 Vector components in 3D

Projecting the components:

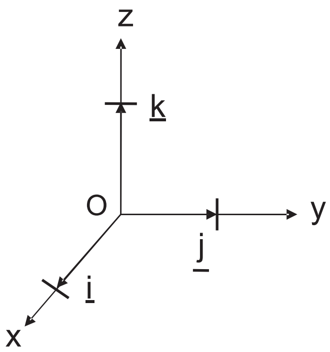
$$\underline{\mathbf{p}} = (p_x, p_y, p_z)$$

- x-component
  $p_x = |\underline{\mathbf{p}}| \sin(\theta) \cos(\phi)$
- y-component
  $p_y = |\underline{\mathbf{p}}| \sin(\theta) \sin(\phi)$
- z-component
  $p_z = |\underline{\mathbf{p}}| \cos(\theta)$
- Magnitude $|\underline{\mathbf{p}}| = \sqrt{(}p_x^2 + p_y^2 + p_z^2)$
- Direction $\tan(\phi) = (p_y/p_x)$
  $\cos(\theta) = (p_z/|\underline{\mathbf{p}}|)$



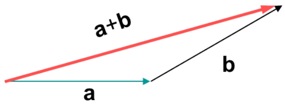
1.4.2 Unit vectors

- A unit vector is a vector with magnitude equal to one.
- e.g. Three unit vectors defined by orthogonal components of the Cartesian coordinate system:
  - $\underline{\mathbf{i}}$ = (1,0,0), obviously $|\underline{\mathbf{i}}| = 1$
  - $\underline{\mathbf{j}}$ = (0,1,0), $|\underline{\mathbf{j}}| = 1$
  - $\underline{\mathbf{k}}$ = (0,0,1), $|\underline{\mathbf{k}}| = 1$
- A unit vector in the direction of general vector $\underline{\mathbf{a}}$ is written $\hat{\underline{\mathbf{a}}} = \underline{\mathbf{a}}/|\underline{\mathbf{a}}|$
- $\underline{\mathbf{a}}$ is written in terms of unit vectors $\underline{\mathbf{a}} = a_x\underline{\mathbf{i}} + a_y\underline{\mathbf{j}} + a_z\underline{\mathbf{k}}$



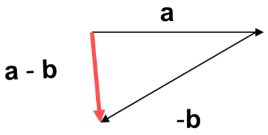
Sum of two vectors

- To calculate the sum of two vectors
  $\underline{\mathbf{c}} = \underline{\mathbf{a}} + \underline{\mathbf{b}}$
  Triangle rule: Put the second vector nose to tail with the first and the resultant is the vector sum.
- $\underline{\mathbf{c}} = \underline{\mathbf{a}} + \underline{\mathbf{b}}$: in (x, y, z) components
  $(c_x, c_y, c_z) = (a_x + b_x, a_y + b_y, a_z + b_z)$
- Alternatively $\underline{\mathbf{c}} = \underline{\mathbf{a}} + \underline{\mathbf{b}}$
  $c_x\underline{\mathbf{i}} + c_y\underline{\mathbf{j}} + c_z\underline{\mathbf{k}} =$
  $(a_x + b_x)\underline{\mathbf{i}} + (a_y + b_y)\underline{\mathbf{j}} + (a_z + b_z)\underline{\mathbf{k}}$



Vector algebra laws

- $\underline{\mathbf{a}} + \underline{\mathbf{b}} = \underline{\mathbf{b}} + \underline{\mathbf{a}}$: commutative law
- $\underline{\mathbf{a}} + (\underline{\mathbf{b}} + \underline{\mathbf{c}}) = (\underline{\mathbf{a}} + \underline{\mathbf{b}}) + \underline{\mathbf{c}}$: associative law
- Can treat vector equations in same way as ordinary algebra
  $\underline{\mathbf{a}} + \underline{\mathbf{b}} = \underline{\mathbf{c}} \quad \Rightarrow \quad \underline{\mathbf{a}} = \underline{\mathbf{c}} - \underline{\mathbf{b}}$
- Note that vector $-\underline{\mathbf{b}}$ is equal in magnitude to $\underline{\mathbf{b}}$ but in the opposite direction.
  so $\underline{\mathbf{a}} - \underline{\mathbf{b}} = \underline{\mathbf{a}} + (-\underline{\mathbf{b}})$



a - b = a + (-b)

Multiplication of a vector by a scalar

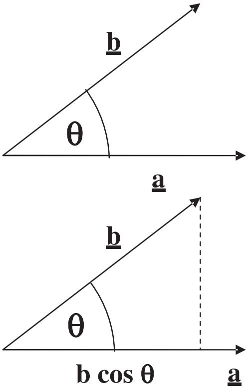
- This gives a vector in the same direction as the original but of proportional magnitude.
- For any scalars α and β and vectors $\underline{\mathbf{a}}$ and $\underline{\mathbf{b}}$
  - $(\alpha\beta)\ \underline{\mathbf{a}} = \alpha(\beta\ \underline{\mathbf{a}}\ ) = \beta(\alpha\ \underline{\mathbf{a}}\ ) = \underline{\mathbf{a}}\ (\alpha\beta)$: associative & commutative
  - $(\alpha + \beta)\underline{\mathbf{a}} = \alpha\underline{\mathbf{a}} + \beta\underline{\mathbf{a}}$: distributive
  - $\alpha(\underline{\mathbf{a}} + \underline{\mathbf{b}}) = \alpha\underline{\mathbf{a}} + \alpha\underline{\mathbf{b}}$

2.1.1 The scalar (dot) product

Scalar (or dot) product definition:
$\underline{\mathbf{a}}.\underline{\mathbf{b}} = |\underline{\mathbf{a}}|.|\underline{\mathbf{b}}|\cos\theta \equiv ab\cos\theta$

(write shorthand $|\underline{\mathbf{a}}| = \mathrm{a}$).

- Scalar product is the magnitude of $\underline{\mathbf{a}}$ multiplied by the projection of $\underline{\mathbf{b}}$ onto $\underline{\mathbf{a}}$.
- Obviously if $\underline{\mathbf{a}}$ is perpendicular to $\underline{\mathbf{b}}$ then $\underline{\mathbf{a}}.\underline{\mathbf{b}} = 0$
- Also $\underline{\mathbf{a}}.\underline{\mathbf{a}} = |a|^2$ (since θ =0°)
  Hence $a = \sqrt{(\underline{\mathbf{a}}.\underline{\mathbf{a}})}$



Properties of scalar product

(i) $\underline{\mathbf{i}}.\underline{\mathbf{i}} = \underline{\mathbf{j}}.\underline{\mathbf{j}} = \underline{\mathbf{k}}.\underline{\mathbf{k}} = 1$ and $\underline{\mathbf{i}}.\underline{\mathbf{j}} = \underline{\mathbf{j}}.\underline{\mathbf{k}} = \underline{\mathbf{k}}.\underline{\mathbf{i}} = 0$

(ii) This leads to $\underline{\mathbf{a}}.\underline{\mathbf{b}} = (a_x\underline{\mathbf{i}} + a_y\underline{\mathbf{j}} + a_z\underline{\mathbf{k}}).(b_x\underline{\mathbf{i}} + b_y\underline{\mathbf{j}} + b_z\underline{\mathbf{k}})$
= $a_x b_x + a_y b_y + a_z b_z$

iii) $\underline{\mathbf{a}}.\underline{\mathbf{b}} = \underline{\mathbf{b}}.\underline{\mathbf{a}}$: commutative
$\underline{\mathbf{a}}.(\underline{\mathbf{b}} + \underline{\mathbf{c}}) = \underline{\mathbf{a}}.\underline{\mathbf{b}} + \underline{\mathbf{a}}.\underline{\mathbf{c}}$: distributive

(iv) Parentheses are important
Note $(\underline{\mathbf{u}}.\underline{\mathbf{v}})\ \underline{\mathbf{w}} \neq \underline{\mathbf{u}}\ (\underline{\mathbf{v}}.\underline{\mathbf{w}})$ because one is a vector along $\underline{\hat{\mathbf{w}}}$, the other is along $\underline{\hat{\mathbf{u}}}$.

2.1.2 The vector (cross) product

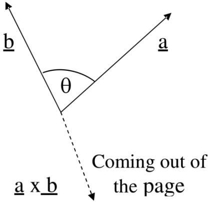
Vector (or cross) product of two vectors, definition:

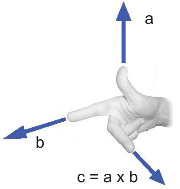
$$\underline{\mathbf{a}} \times \underline{\mathbf{b}} = |\underline{\mathbf{a}}||\underline{\mathbf{b}}| \sin\theta \ \underline{\hat{\mathbf{n}}}$$

where $\underline{\hat{\mathbf{n}}}$ is a *unit vector* in a direction *perpendicular* to both $\underline{\mathbf{a}}$ and $\underline{\mathbf{b}}$.

To get direction of $\underline{\mathbf{a}} \times \underline{\mathbf{b}}$ use right hand rule:

- i) Make a set of directions with your *right hand*$\rightarrow$ thumb & first index finger, and with middle finger positioned perpendicular to plane of both
- ii) Point your thumb along the first vector $\underline{\mathbf{a}}$
- iii) Point your 1st index finger along $\underline{\mathbf{b}}$, making the smallest possible angle to $\underline{\mathbf{a}}$
- iv) The direction of the middle finger gives the direction of $\underline{\mathbf{a}} \times \underline{\mathbf{b}}$.





Properties of vector product

- $\underline{\mathbf{i}} \times \underline{\mathbf{j}} = \underline{\mathbf{k}} \quad ; \quad \underline{\mathbf{j}} \times \underline{\mathbf{k}} = \underline{\mathbf{i}} \quad ; \quad \underline{\mathbf{k}} \times \underline{\mathbf{i}} = \underline{\mathbf{j}} \quad ; \quad \underline{\mathbf{i}} \times \underline{\mathbf{i}} = 0$ etc.
- $(\underline{\mathbf{a}} + \underline{\mathbf{b}}) \times \underline{\mathbf{c}} = (\underline{\mathbf{a}} \times \underline{\mathbf{c}}) + (\underline{\mathbf{b}} \times \underline{\mathbf{c}})$: distributive
- $\underline{\mathbf{a}} \times \underline{\mathbf{b}} = -\underline{\mathbf{b}} \times \underline{\mathbf{a}}$: NON-commutative
- $(\underline{\mathbf{a}} \times \underline{\mathbf{b}}) \times \underline{\mathbf{c}} \neq \underline{\mathbf{a}} \times (\underline{\mathbf{b}} \times \underline{\mathbf{c}})$: NON-associative
- If m is a scalar,
  $m(\underline{\mathbf{a}} \times \underline{\mathbf{b}}) = (m\underline{\mathbf{a}}) \times \underline{\mathbf{b}} = \underline{\mathbf{a}} \times (m\underline{\mathbf{b}}) = (\underline{\mathbf{a}} \times \underline{\mathbf{b}})m$
- $\underline{\mathbf{a}} \times \underline{\mathbf{b}} = 0$ if vectors are parallel (0^o)
  i.e $\quad \underline{\mathbf{a}} \times \underline{\mathbf{a}} = 0$

Vector product in components

Cross product written out in components:

- $\underline{\mathbf{a}} \times \underline{\mathbf{b}} = (a_x, a_y, a_z) \times (b_x, b_y, b_z)$
  $= (a_x\underline{\mathbf{i}} + a_y\underline{\mathbf{j}} + a_z\underline{\mathbf{k}}) \times (b_x\underline{\mathbf{i}} + b_y\underline{\mathbf{j}} + b_z\underline{\mathbf{k}})$
- Since $\underline{\mathbf{i}} \times \underline{\mathbf{i}} = \underline{\mathbf{j}} \times \underline{\mathbf{j}} = \underline{\mathbf{k}} \times \underline{\mathbf{k}} = 0$ and $\underline{\mathbf{i}} \times \underline{\mathbf{j}} = \underline{\mathbf{k}}$ etc.
- $\underline{\mathbf{a}} \times \underline{\mathbf{b}} = (a_y b_z - a_z b_y)\underline{\mathbf{i}} - (a_x b_z - a_z b_x)\underline{\mathbf{j}} + (a_x b_y - a_y b_x)\underline{\mathbf{k}}$

This is much easier when we write in *determinant* form:

$$\underline{\mathbf{a}} \times \underline{\mathbf{b}} = \begin{vmatrix} \underline{\mathbf{i}} & \underline{\mathbf{j}} & \underline{\mathbf{k}} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} \tag{1}$$

2.1.3 Examples of scalar & vector products in mechanics

- a) Scalar product

  Work done on a body by a force through distance $\underline{\mathbf{dr}}$ from position 1 to 2

  $$W_{12} = \int_1^2 \underline{\mathbf{F}}.\underline{\mathbf{dr}}$$

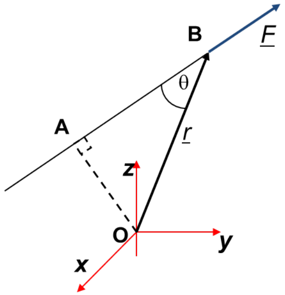
  Only the component of force parallel to the line of displacement does work.

- b) Vector product

  A *torque* about O due to a force $\underline{\mathbf{F}}$ acting at B :

  $$\underline{\tau} = \underline{\mathbf{r}} \times \underline{\mathbf{F}}$$

  Torque is a vector with direction perpendicular to both $\underline{\mathbf{r}}$ and $\underline{\mathbf{F}}$, magnitude of $|\underline{\mathbf{r}}||\underline{\mathbf{F}}| \sin \theta$.



2.2 Differentiation of vectors

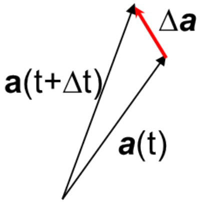
Notation: a dot above the function indicates derivative wrt time.
A “dash” indicates derivative wrt a spatial coordinate.

$$\dot{y} \equiv \frac{dy}{dt} \qquad y' \equiv \frac{dy}{dx}$$

$$\dot{\underline{\mathbf{a}}} = \lim_{\Delta t \to 0} \frac{\underline{\mathbf{a}}(t+\Delta t) - \underline{\mathbf{a}}(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta \underline{\mathbf{a}}}{\Delta t}$$

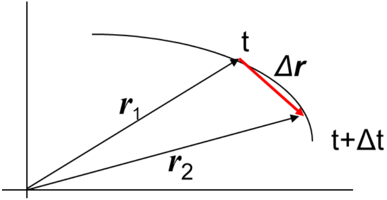
$$\underline{\mathbf{a}}(t) = a_x(t)\underline{\mathbf{i}} + a_y(t)\underline{\mathbf{j}} + a_z(t)\underline{\mathbf{k}}$$

$$\dot{\underline{\mathbf{a}}} = \lim_{\Delta t \to 0} \left(\frac{a_x(t+\Delta t) - a_x(t)}{\Delta t}\, \underline{\mathbf{i}} + \ \ldots \right)$$



Hence $\dot{\underline{\mathbf{a}}} = \dot{a}_x\underline{\mathbf{i}} + \dot{a}_y\underline{\mathbf{j}} + \dot{a}_z\underline{\mathbf{k}}$

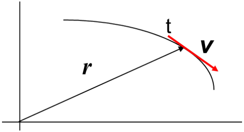
2.2.1 Vector velocity



- $\Delta \underline{\mathbf{r}} = \underline{\mathbf{r}}_2 - \underline{\mathbf{r}}_1$

  $$\underline{\mathbf{v}} = \lim_{\Delta t \to 0} \frac{\Delta \underline{\mathbf{r}}}{\Delta t}$$

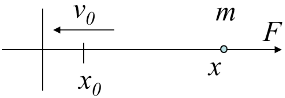
- Velocity at any point is tangent to the path at that point
- $\underline{\mathbf{v}} = \frac{d\underline{\mathbf{r}}}{dt} = \dot{\underline{\mathbf{r}}}$



In one dimension:
Abandon vector notation and simply write $v = \frac{dx}{dt} = \dot{x}$,
($+v$ in $+x$ direction, $-v$ in $-x$ direction).

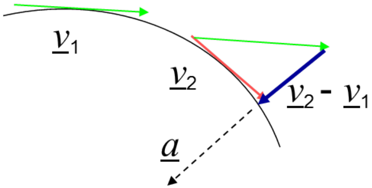
Example - 1D motion

A body has velocity $v_0 = -15\,ms^{-1}$ at position $x_0 = 20m$ and has a time-dependent acceleration $a(t) = 6t - 4\,[ms^{-2}]$. Find the value of x for which the body instantaneously comes to rest.



- $a(t) = 6t - 4\ [ms^{-2}]\ ;\ x_0 = 20\,m\ ;\ v_0 = -15\,ms^{-1}$
- $\dot{v} = 6t - 4 \quad \rightarrow \quad v = \int a(t)dt = 3t^2 - 4t + c$
- At $t = 0, v = -15\ ms^{-1} \quad \rightarrow \quad c = -15\ ms^{-1}$
- $v = 3t^2 - 4t - 15$
- $v = 0$ for $3t^2 - 4t - 15 = 3(t-3)(t+\frac{5}{3}) = 0$
  $\rightarrow t = 3s$ (also $-\frac{5}{3}s$)
- $x = \int v(t)dt = t^3 - 2t^2 - 15t + c' \rightarrow x = 20\,m$ at $t = 0, c' = 20\,m$
- $x(t) = 27 - 18 - 45 + 20 = -16\,\mathrm{m}$

2.2.2 Vector acceleration

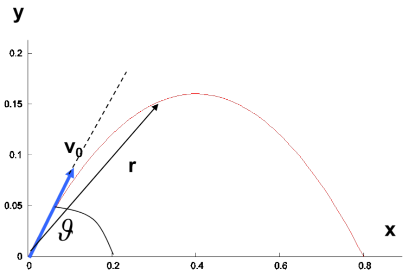


- $\Delta \underline{\mathbf{v}} = \underline{\mathbf{v}}_2 - \underline{\mathbf{v}}_1$

$$\underline{\mathbf{a}} = \lim_{\Delta t \to 0} \frac{\Delta \underline{\mathbf{v}}}{\Delta t} = \dot{\underline{\mathbf{v}}} = \ddot{\underline{\mathbf{r}}}$$

In one dimension:
Abandon vector notation and simply write $a = \frac{dv}{dt} = \dot{v} = \ddot{x}$,
($+a$ in $+x$ direction, $-a$ in $-x$ direction).

Example: constant acceleration - projectile motion in 2D



- $\underline{\mathbf{a}} = \frac{d\underline{\mathbf{v}}}{dt} = \text{constant}$
- $\underline{\mathbf{r}} = 0$ at $t = 0$
- $\int_{v_0}^{v} d\underline{\mathbf{v}} = \int_0^t \underline{\mathbf{a}} dt$

  $\rightarrow \underline{\mathbf{v}} = \underline{\mathbf{v}}_0 + \underline{\mathbf{a}} t \quad \rightarrow \underline{\mathbf{v}} = \frac{d\underline{\mathbf{r}}}{dt}$
- $\int_0^r d\underline{\mathbf{r}} = \int_0^t (\underline{\mathbf{v}}_0 + \underline{\mathbf{a}} t) dt$

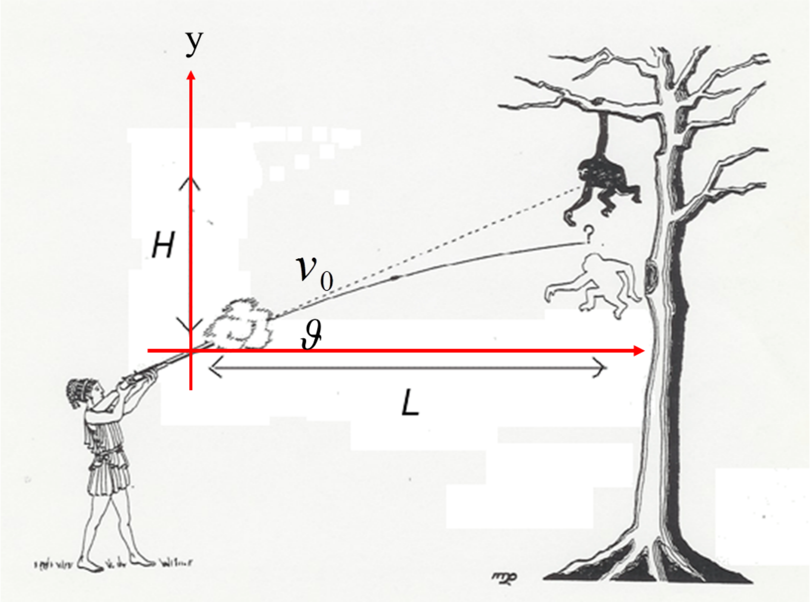
  $\rightarrow \underline{\mathbf{r}} = \underline{\mathbf{v}}_0 t + \frac{1}{2} \underline{\mathbf{a}} t^2$

Under gravity: $\underline{\mathbf{a}} = -g\underline{\hat{\mathbf{y}}} \quad \rightarrow \quad a_x = 0; \quad a_y = -g$

- $v_x = v_0 \cos\theta$
- $v_y = v_0 \sin\theta - gt$
- $x = (v_0 \cos\theta) t$
- $y = (v_0 \sin\theta) t - \frac{1}{2} g t^2$

Trajectory: $\quad y = (\tan\theta) x - \frac{g}{2v_0^2} (\sec^2\theta) x^2$

The monkey and the hunter



3.1 Dimensional analysis

- A useful method for determining the units of a variable in an equation
- Useful for checking the correctness of an equation which you have derived after some algebraic manipulation. Dimensions need to be correct !
- Determining the form of an equation itself

Most physical quantities can be expressed in terms of combinations of basic dimensions. These are certainly not unique :

- mass (M)
- length (L)
- time (T)
- electric charge (Q)
- temperature (θ)

Note: The term "dimension" is not quite the same as "unit", but obviously closely related.

Quantity	Unit		Dimension
Frequency	Hertz (Hz)	= (cycles) s^{-1}	T^{-1}
Force	Newton (N)	= kg m s^{-2}	MLT^{-2}
Energy	Joule (J)	= N m = kg m$^2 s^{-2}$	ML^2T^{-2}
Power	Watt (W)	= J s^{-1} = kg m$^2 s^{-3}$	ML^2T^{-3}
Current	Ampere (A)	= Cs^{-1}	QT^{-1}
EMF	Volt (V)	= Nm C^{-1} = kg m$^2 s^{-2} C^{-1}$	ML^{2}T^{-2}Q^{-1}

Dimensional analysis is best illustrated with examples.

3.1.1 The period of a pendulum

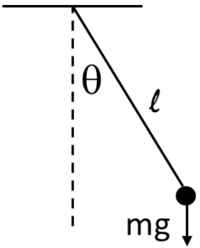
How does the period of a pendulum depend on its length?

- Variables: period P, mass m, length l, acceleration due to gravity g
- Guess the form: let $P = k\, m^a \ell^b g^c$

  (k is a dimensionless constant)
- $T^1 = M^a L^b (LT^{-2})^c = M^a L^{b+c} T^{-2c}$
- Compare terms:

  $a = 0,\ b + c = 0,\ -2c = 1$

  $\rightarrow \quad c = -1/2,\ b = 1/2$



$$P = k\sqrt{\frac{\ell}{g}}$$

We know that $P = 2\pi\sqrt{\frac{\ell}{g}}$: we obtained this form using dimensions and without using equation of motion: IMPRESSIVE !

3.1.2 Kepler's third law

How does the period of an orbiting mass depend on its radius?

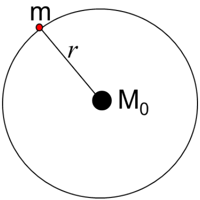
- Variables: period P, central mass M_0, orbit radius r, Gravitational constant G
- Guess the form: let $P = k\, M_0^a r^b G^c$

  (k is a dimensionless constant)
- Dimensions of $G \to (MLT^{-2}).L^2 M^{-2}$
- $T^1 = M^a L^b (M^{-1} L^3 T^{-2})^c$

  $= M^{(a-c)} L^{b+3c} T^{-2c}$
- Compare terms:

  $a - c = 0,\ b + 3c = 0,\ -2c = 1$

  $\to \quad a = -1/2,\ c = -1/2,\ b = 3/2$



- $\frac{GmM_0}{r^2} = \frac{mv^2}{r}$
- $v = \frac{2\pi r}{P}$
- $P^2 = \frac{4\pi^2}{GM_0} r^3$

$\to \quad k^2 = 4\pi^2$

$$P = k\, M_0^{-1/2} r^{3/2} G^{-1/2} \to \quad P^2 = \frac{k^2}{GM_0} r^3$$

3.1.3 The range of a cannon ball

A cannon ball is fired with V_y upwards and V_x horizontally, assume no air resistance.

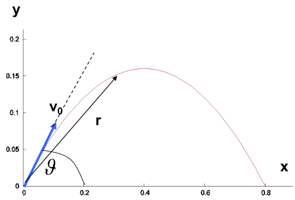
- Variables: V_x, V_y, distance travelled along x (range) R, acceleration due to gravity g
- First with no use of directed length dimensions
- Let $R = kV_x^a V_y^b g^c$.

  (k is a dimensionless constant)
- Dimensionally $L = (L/T)^{a+b}(L/T^2)^c$
- Compare terms:

  $a + b + c = 1$ and $a + b + 2c = 0$, which leaves one exponent undetermined.
- Now use directed length dimensions, then V_x will be dimensioned as L_x/T, V_y as L_y/T, R as L_x and g as L_y/T^2
- The dimensional equation becomes: $L_x = (L_x/T)^a (L_y/T)^b (L_y/T^2)^c$

  $\rightarrow a = 1, b = 1$ and $c = -1$.

$$R = k\frac{v_x v_y}{g}$$



- $x = v_x t$
- $y = v_y t - \frac{1}{2}gt^2$

  $= 0$

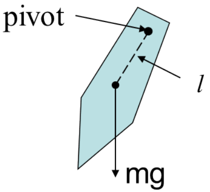
  $\rightarrow t = \frac{2v_y}{g}$
- $x = \frac{2v_x v_y}{g}$

3.1.4 Example of limitations of the method

- Let $y = f(x_1,\ x_2, \dots x_n)$ where $x_1,\ x_2, \dots x_n$ have independent dimensions
- However in general $y = (x_1^a\, x_2^b \dots x^n)\, \phi(u_1, \dots u_k)$ where u_i are dimensionless variables

Extend to how the period of a *rigid* pendulum depends on length pivot to CM.

- In actual fact $P \equiv P(g, \ell, m, \mathtt{I})$ where $\mathtt{I}$ is the moment of inertia
- $[\mathtt{I}] = ML^2 \ \rightarrow$ can define $u = \frac{\mathtt{I}}{m\ell^2}$



$$T = \sqrt{\frac{\ell}{g}}\, \phi(u)$$

i.e. Equation is not reproduced

$$T = 2\pi\sqrt{\frac{\mathtt{I}}{mg\ell}}$$

3.2 Newton's Laws of motion

- NI: Every body continues in a state of rest or in uniform motion (constant velocity in straight line) unless acted upon by an external force.

- NII: The rate of change of momentum is equal to the applied force; where the momentum is defined as the product of mass and velocity ($\underline{\mathbf{p}} = m\underline{\mathbf{v}}$). [i.e. the applied force $\underline{\mathbf{F}}$ on a body is equal to its mass m multiplied by its acceleration $\underline{\mathbf{a}}$.]

- NIII: When one body exerts a force on a second body, the second body simultaneously exerts a force equal in magnitude and opposite in direction on the first body [i.e. action and reaction forces are equal in magnitude and opposite in direction.]

3.3 Frames of reference

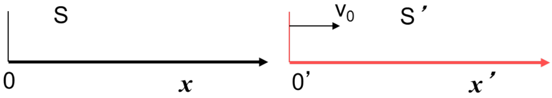
- A frame of reference is an environment which is used to observe an event or the motion of a particle.
- A coordinate system is associated with the frame to observe the event (eg the body's location over time).
- The observer is equipped with measuring tools (eg rulers and clocks) to measure the positions and times of events.
- In classical mechanics, time intervals between events is the same in all reference frames (time is absolute).
- In relativity, we will need to use space-time frames.
- A reference frame in which NI is satisfied is called an *inertial reference frame*.

Inertial reference frames

A frame in which Newton's first law is satisfied:

- Deep space
- The Earth? [Only in circumstances where we can ignore gravity & the spin of the Earth.]

Principle of Relativity : The laws of Physics are the same in all inertial frames of reference.

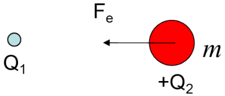


At $t = 0, x = 0, x' = 0$ and S and S' are coincident.

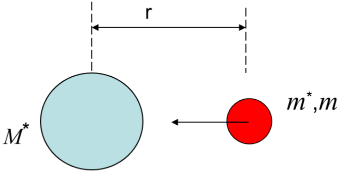
Galilean Transformation of coordinates:

- $x' = x - v_0 t, \;\; y' = y, \;\; z' = z, \;\; t' = t$
- Velocity of a body v in S; velocity measured in S' $\quad v' = v - v_0$
- Acceleration measured in S' $\quad a' = a$
- Hence $F' = F$ (consistent with the principle of relativity)

3.4 The Principle of Equivalence



$$F_e = ma \qquad \left(F_e = k\frac{Q_1 Q_2}{r^2} \right)$$



$$F_g = ma \qquad \left(F_g = G\frac{M^* m^*}{r^2} \right)$$

- The Principle of Equivalence dictates that $m = m^*$.
- Inertial mass = Gravitational mass
- This may seem obvious, but it was not an original postulate of Newton

4.1 Newton's Second Law

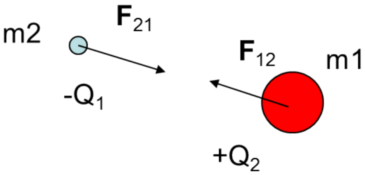
The rate of change of momentum of a body is equal to the applied force on the body.

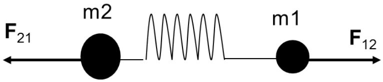
- $\underline{\mathbf{F}} = \frac{d\underline{\mathbf{p}}}{dt} = m\underline{\mathbf{a}}$ where $\underline{\mathbf{p}} = m\underline{\mathbf{v}}$
- In components: $(F_x, F_y, F_z) = m(a_x, a_y, a_z)$
- Assuming constant mass, we can define *the equation of motion* in 1D : $F = \frac{d(mv)}{dt} = m\frac{d^2x}{dt^2}$
- We require two initial conditions *for a unique solution*: e.g. $v = v_0$ at $t = 0$ and $x = x_0$ at $t = 0$

We shall later solve the EOM for three examples:
(i) F = constant, (ii) $F \propto -v$, (iii) $F \propto -x$

4.2 Newton's Third Law

Action and reaction forces are equal in magnitude and opposite in direction.



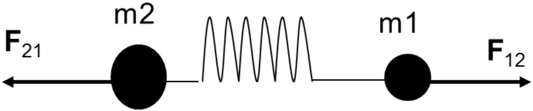


Compressed spring

Electrostatic interaction

$$\underline{\mathbf{F}}_{12} = -\underline{\mathbf{F}}_{21}$$

Conservation of momentum



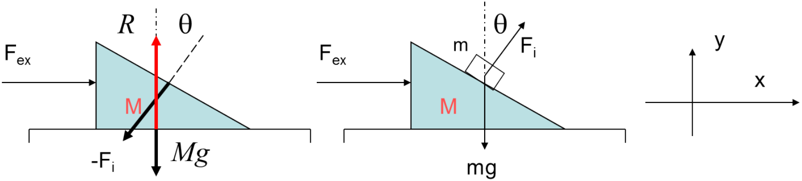
Compressed spring

- $\underline{\mathbf{F}}_{12} = m_1 \underline{\mathbf{a}}_1 = \frac{d\underline{\mathbf{P}}_1}{dt}$ and $\underline{\mathbf{F}}_{21} = m_2 \underline{\mathbf{a}}_2 = \frac{d\underline{\mathbf{P}}_2}{dt}$
- $\underline{\mathbf{F}}_{12} + \underline{\mathbf{F}}_{21} = \frac{d}{dt}(\underline{\mathbf{P}}_1 + \underline{\mathbf{P}}_2) = 0$ (Newton III)
- Therefore $(\underline{\mathbf{P}}_1 + \underline{\mathbf{P}}_2)$ = constant

In an isolated system, the total momentum is conserved.

Newton II : Example 1. E.O.M. under constant force

How fast should we accelerate the triangular wedge to keep the block *m* stationary on the wedge?



Forces on wedge:

- Horizontal: $F_{ex} - F_i \sin\theta = MA_x$
- Vertical: $R - F_i \cos\theta - Mg = 0$

Forces on block:

- Horizontal: $F_i \sin\theta = ma_x$
- Vertical: $F_i \cos\theta - mg = ma_y$

For block to remain at the same place $A_x = a_x$ and $a_y = 0$

- $F_i = \frac{mg}{\cos\theta}$ and $a_x = g\tan\theta = A_x$
- Hence $F_{ex} = Mg\tan\theta + mg\tan\theta = (m+M)g\tan\theta$

Example constant force, continued

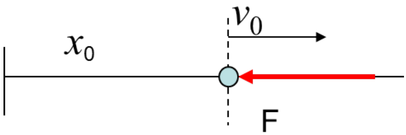
What is the internal force that the blocks apply on each other and the reaction force by the ground on M?

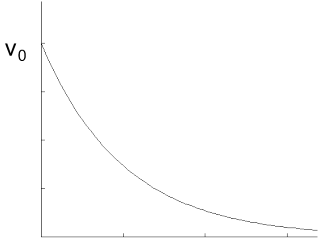
From before:

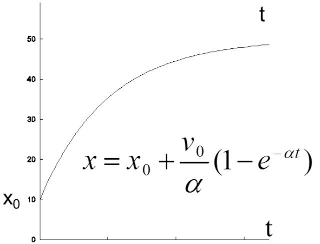
- $F_i = \frac{mg}{\cos\theta}$
- $R - F_i \cos\theta - Mg = 0$
- Hence: $R = F_i \cos\theta + Mg = (m + M)g$

Example 2. Force proportional to velocity

Solve the equation of motion for the case $F = -\beta v \;\; (\beta > 0)$ with $x = x_0$ and $v = v_0$ at $t = 0$



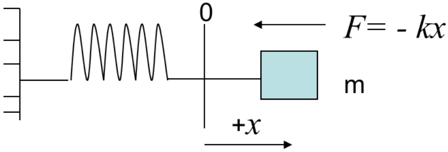




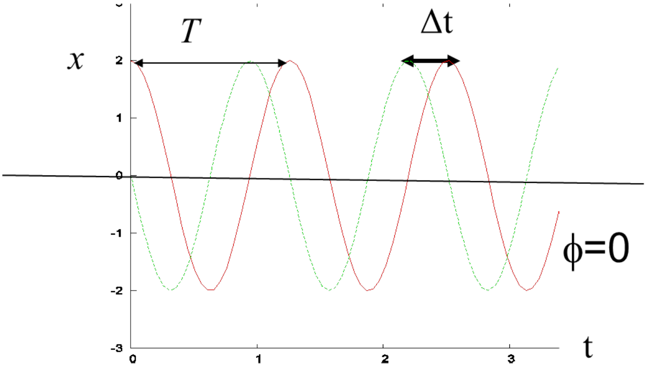
- $m\frac{dv}{dt} = -\beta v$
- $\frac{dv}{dt} = -\alpha v$ where $\alpha = \frac{\beta}{m}$
- $\int_{v_0}^{v} \frac{dv}{v} = -\alpha \int_0^t dt \;\; \rightarrow \;\; v = v_0 e^{-\alpha t}$
- $v = \frac{dx}{dt} \rightarrow \int_{x_0}^{x} dx = \int_0^t v dt = \int_0^t v_0 e^{-\alpha t} dt$
- $x - x_0 = -\frac{v_0}{\alpha} e^{-\alpha t} + \frac{v_0}{\alpha}$
- $x = x_0 + \frac{v_0}{\alpha}(1 - e^{-\alpha t})$
- When $t \rightarrow \infty, \;\; x \rightarrow x_0 + \frac{v_0}{\alpha}$

Example 3. Force proportional to position: simple harmonic oscillator

Solving the equation of motion for the case $F = m\frac{d^2x}{dt^2} = -kx$



- $m\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$; trial solution $x = A\cos\omega t + B\sin\omega t$
  $\rightarrow \dot{x} = -A\omega\sin\omega t + B\omega\cos\omega t$; $\ddot{x} = -A\omega^2\cos\omega t - B\omega^2\sin\omega t$
- $\ddot{x} = -\omega^2 x \rightarrow \omega^2 = \frac{k}{m}$
- Alternatively $x = x_0\cos(\omega t + \phi)$ (or $x = x_0 Re[e^{i(\alpha t+\phi)}]$)
- Expand : $x = x_0(\cos(\omega t)\cos\phi - \sin(\omega t)\sin\phi)$
  $A = x_0\cos\phi$; $B = -x_0\sin\phi \rightarrow x_0^2 = A^2 + B^2$; $\tan\phi = -B/A$
- x_0 = amplitude, ϕ = phase, ω = angular frequency ($T = \frac{2\pi}{\omega}$)



- $x = x_0 \cos(\omega t + \phi)$
- $\omega = \sqrt{\frac{k}{m}}$
- $\phi = \omega \Delta t$

4.3 Energy conservation in one dimension

Work done on a body by a force F

- $W = \int_{x_1}^{x_2} F(x)dx = m \int_{x_1}^{x_2} \frac{dv}{dt} dx$
- We can write: $\frac{dv}{dt} dx = \frac{dx}{dt} dv = vdv$

  hence $\int_{x_1}^{x_2} F(x)dx = m \int_{v_1}^{v_2} vdv = \frac{1}{2} m(v_2^2 - v_1^2) = T_2 - T_1$
- Now introduce an arbitrary reference point x_0

  $\int_{x_1}^{x_2} Fdx = \int_{x_0}^{x_2} Fdx - \int_{x_0}^{x_1} Fdx$ defines a *conservative* force

  hence $T_2 + [-\int_{x_0}^{x_2} Fdx] = T_1 + [-\int_{x_0}^{x_1} Fdx]$
- We define the *potential energy* $U(x)$ at a point x :

  $U(x) - U(x_0) = -\int_{x_0}^{x} Fdx$ and hence

  $T_2 + U_2 = T_1 + U_1$ (total energy PE + KE conserved)
- Note the minus sign. The potential energy (relative to a reference point) is always the *negative* of the work done by the force $\rightarrow$ $F(x) = -\frac{dU}{dx}$

5.1 Conservative forces

$$W_{ab} = \int_a^b \underline{\mathbf{F}} \,.\, d\underline{\mathbf{r}} = U(a) - U(b)$$

For a conservative field of force, the work done depends only on the initial and final positions of the particle independent of the path.

The conditions for a conservative force (*all equivalent*) are:

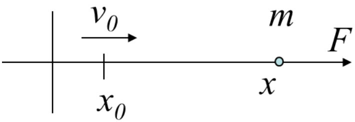
- The force is derived from a (scalar) potential function: $\underline{\mathbf{F}}(\underline{\mathbf{r}}) = -\nabla U \quad \rightarrow \quad F(x) = -\frac{dU}{dx}$ etc.
- There is zero net work by the force when moving a particle around any closed path: $W = \oint_c \underline{\mathbf{F}} \,.\, d\underline{\mathbf{r}} = 0$
- In equivalent vector notation $\underline{\nabla} \times \underline{\mathbf{F}} = 0$

For *any* force: $W_{ab} = \frac{1}{2}mv_b^2 - \frac{1}{2}mv_a^2$

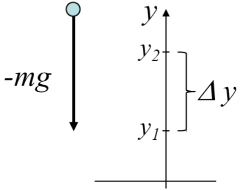
Only for a *conservative* force: $W_{ab} = U(a) - U(b)$

Conservative force: example 1. Constant acceleration

Consider a particle moving under constant force (in 1-D).



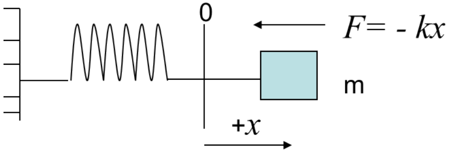
- $F = ma$. Say at $t = 0 \rightarrow x = x_0$ and $v = v_0$
- $T_2 + U_2 = T_1 + U_1$ (the total energy is conserved)
- $\frac{1}{2}mv^2 - (ma)x = \frac{1}{2}mv_0^2 - (ma)x_0 =$ constant
- $v^2 = v_0^2 + 2a(x - x_0)$



- Gravitational potential energy
- $U(\Delta y) = -\int_{y_1}^{y_2} F(y)dy$
- $U(\Delta y) = -\int_{y_1}^{y_2}(-mg)dy = mg(y_2 - y_1)$

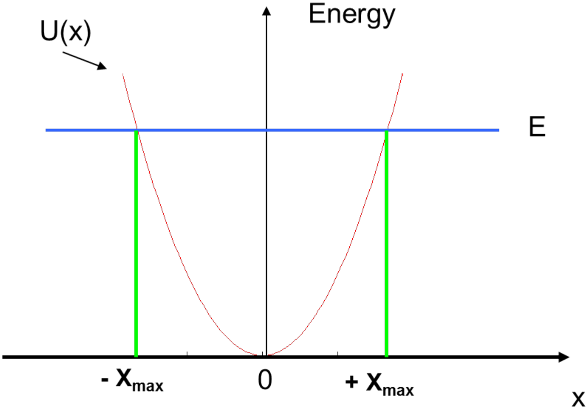
Example 2. Simple harmonic oscillator

Equation of motion: $F = m\frac{d^2x}{dt^2} = -kx$



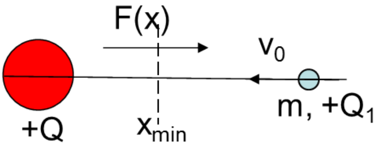
- Potential energy: $U(x) = -\int_0^x F dx = -\int_0^x (-kx) dx = \frac{kx^2}{2}$
- Total energy: $E = T(x) + U(x) = \frac{1}{2}m\dot{x}^2 + \frac{kx^2}{2}$
- Check conservation of energy:
  EOM : $m\ddot{x} + kx = 0 \;\to$ [multiply by $\dot{x}$] $\; m\ddot{x}\dot{x} + kx\dot{x} = 0$
  $\to \; \frac{1}{2}m\frac{d}{dt}(\dot{x}^2) + \frac{1}{2}k\frac{d}{dt}(x^2) = 0$
- Integrate wrt t: $\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = \text{constant} \quad \to \quad$ i.e. energy conserved.

SHM potential energy curve



- $E = U(x) + \frac{1}{2}mv^2$
- The particle can only reach locations x that satisfy $U < E$

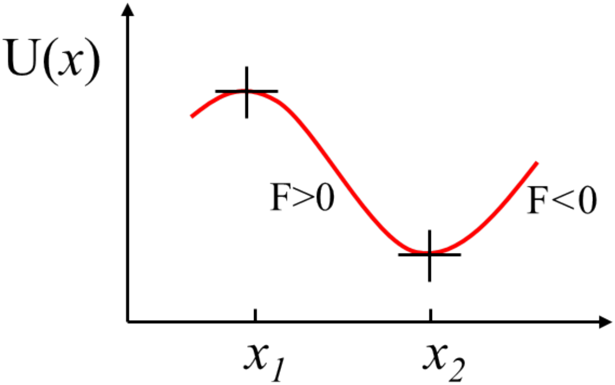
Example 3. Minimum approach of a charge



A particle of mass m and charge $+Q_1$ starts from $x = +\infty$ with velocity v_0. It approaches a fixed charge $+Q$. Calculate its minimum distance of approach x_{min}.

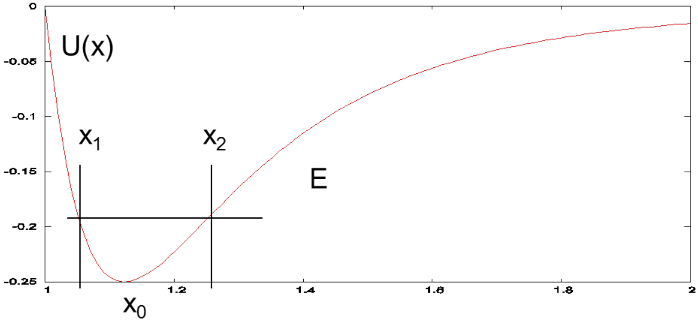
- Force on charge $+Q_1$: $F(x) = +\frac{QQ_1}{4\pi\epsilon_0 x^2}$ (+ve direction)
- Potential energy at point x: $U(x) = -\int_{\infty}^{x} F(x)dx = +\frac{QQ_1}{4\pi\epsilon_0 x}$ (where PE = 0 at $x = \infty$)
- Conservation of energy : $\frac{1}{2}mv_0^2 + 0 = \frac{1}{2}mv^2 + U(x)$
- Min. dist. when $v = 0$: $\frac{1}{2}mv_0^2 = \frac{QQ_1}{4\pi\epsilon_0 x_{min}} \rightarrow x_{min} = \frac{QQ_1}{2\pi m\epsilon_0 v_0^2}$

5.2 Potential with turning points



- U is a maximum: *unstable* equilibrium
- U is a minimum: *stable* equilibrium

5.2.1 Oscillation about stable equilibrium

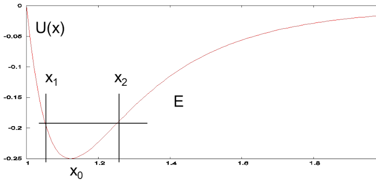


- For SHM : $U(x) = \frac{1}{2}k(x - x_0)^2$
- Taylor expansion about x_0:

$$U(x) = U(x_0) + \underbrace{\left[\frac{dU}{dx}\right]_{x=x_0}}_{=0}(x - x_0) + \frac{1}{2!}\underbrace{\left[\frac{d^2U}{dx^2}\right]_{x=x_0}}_{=k}(x - x_0)^2 + \ldots$$

Example: The Lennard-Jones potential

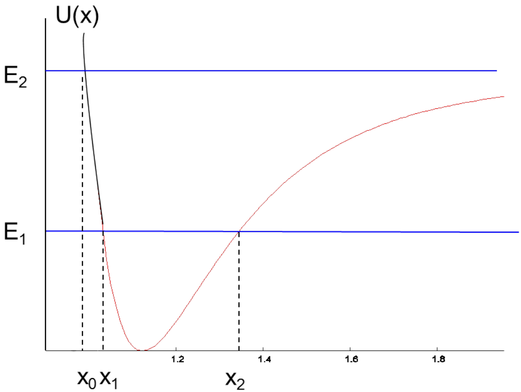
The Lennard-Jones potential describes the potential energy between two atoms in a molecule: $U(x) = \epsilon[(x_0/x)^{12} - 2(x_0/x)^6]$ (ϵ and x_0 are constants and x is the distance between the atoms).



Show that the motion for small displacements about the minimum is simple harmonic and find its frequency.

- $U(x) = U(x_0) + \left[\frac{dU}{dx}\right]_{x=x_0}(x - x_0) + \frac{1}{2!}\left[\frac{d^2U}{dx^2}\right]_{x=x_0}(x - x_0)^2 + \ldots$
- $U(x_0) = \epsilon[(x_0/x)^{12} - 2(x_0/x)^6]_{x=x_0} = -\epsilon$
- $\frac{dU(x)}{dx}|_{x=x_0} = 12\epsilon[-\frac{1}{x_0}(x_0/x)^{13} + \frac{1}{x_0}(x_0/x)^7]_{x=x_0} = 0$ as expected.
- $\frac{d^2U(x)}{dx^2}|_{x=x_0} = \frac{12\epsilon}{x_0^2}[13(x_0/x)^{14} - 7(x_0/x)^8]_{x=x_0} = \frac{72\epsilon}{x_0^2}$
- Hence $U(x) \approx -\epsilon + \frac{72\epsilon}{2!x_0^2}(x - x_0)^2$
- $F(x) = -\frac{dU}{dx} \approx -\frac{1}{2} \times 2(\frac{72\epsilon}{x_0^2})(x - x_0) = -k(x - x_0)$ SHM about x_0
- Angular frequency of small oscillations : $\omega^2 = \frac{k}{m} = \frac{72\epsilon}{mx_0^2}$

5.2.2 Bounded and unbounded potentials



- Bounded motion : $E = E_1$: x constrained $x_1 < x < x_2$
- Unbounded motion : $E = E_2$: x unconstrained at high x $x_0 < x < \infty$

6.1 Lab & CM frames of reference

From hereon we will deal with 2 inertial frames:

- The Laboratory frame: this is the frame where measurements are actually made
- The centre of mass frame: this is the frame where the centre of mass of the system is at rest and where the total momentum of the system is zero

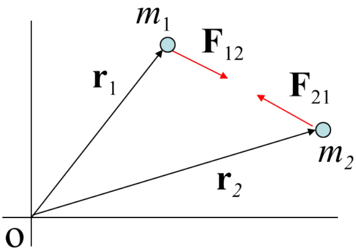
6.2 Internal forces and reduced mass

- Internal forces only:

  $\underline{\mathbf{F}}_{12} = m_1 \ddot{\underline{\mathbf{r}}}_1 \;\; ; \underline{\mathbf{F}}_{21} = m_2 \ddot{\underline{\mathbf{r}}}_2$

  Then $\ddot{\underline{\mathbf{r}}}_2 - \ddot{\underline{\mathbf{r}}}_1 = \frac{\underline{\mathbf{F}}_{21}}{m_2} - \frac{\underline{\mathbf{F}}_{12}}{m_1}$

  NIII : $\underline{\mathbf{F}}_{21} = -\underline{\mathbf{F}}_{12} = \underline{\mathbf{F}}_{int}$



- Define $\underline{\mathbf{r}} = \underline{\mathbf{r}}_2 - \underline{\mathbf{r}}_1 \rightarrow \dot{\underline{\mathbf{r}}} = \dot{\underline{\mathbf{r}}}_2 - \dot{\underline{\mathbf{r}}}_1$
- $\underline{\mathbf{F}}_{int} \left(\frac{1}{m_1} + \frac{1}{m_2} \right) = \ddot{\underline{\mathbf{r}}}$
- Define $\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2} \quad \rightarrow \quad \underline{\mathbf{F}}_{int} = \mu \ddot{\underline{\mathbf{r}}}$

  $\mu = \frac{m_1 m_2}{m_1 + m_2}$ is the *reduced mass* of the system

This defines the equation of motion of the relative motion of the particles under internal forces, with position vector $\underline{\mathbf{r}}$ & mass μ

6.3 The Centre of Mass

The centre of mass (CM) is the point where the mass-weighted position vectors (moments) relative to the point sum to zero ; the CM is the mean location of a distribution of mass in space.

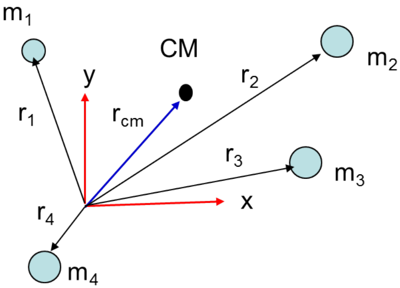
- Take a system of n particles, each with mass m_i located at positions $\underline{\mathbf{r_i}}$, the position vector of the CM is defined by:

  $\sum_{i=1}^{n} m_i(\underline{\mathbf{r_i}} - \underline{\mathbf{r_{cm}}}) = 0$

- Solve for $\underline{\mathbf{r_{cm}}}$:

  $\underline{\mathbf{r_{cm}}} = \frac{1}{M}\sum_{i=1}^{n} m_i \, \underline{\mathbf{r_i}}$

  where $M = \sum_{i=1}^{n} m_i$

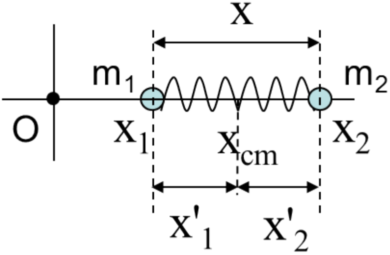


Example : SHM of two connected masses in 1D

SHM between two masses m_1 and m_2 connected by a spring

- $x = x_2 - x_1$; Natural length L
- $F_{int} = -k\,(x - L) = \mu\,\ddot{x}$

  ($\mu = \frac{m_1 m_2}{m_1 + m_2}$ = reduced mass)
- $\ddot{x} + \frac{k}{\mu}(x - L) = 0$



Solution: $x = x_0 \cos(\omega t + \phi) + L$

where $\omega = \sqrt{\frac{k}{\mu}}$

With respect to the CM:

- $x_{CM} = \frac{m_1 x_1 + m_2 x_2}{M}$ where $M = m_1 + m_2$
- $x_1' = x_1 - x_{cm} = \frac{Mx_1 - m_1 x_1 - m_2 x_2}{M} = -\frac{m_2 x}{M}$
- $x_2' = x_2 - x_{cm} = \frac{Mx_2 - m_1 x_1 - m_2 x_2}{M} = \frac{m_1 x}{M}$

Eg. take $m_1 = m_2 = m \;\; \rightarrow \;\; \omega = \sqrt{\frac{2k}{m}}$; $\;\; x_1' = -\frac{1}{2}x, \;\; x_2' = \frac{1}{2}x$

6.3.1 CM of a continuous volume

If the mass distribution is continuous with density $\rho(\underline{\mathbf{r}})$ inside a volume V, then:

- $\sum_{i=1}^{n} m_i(\underline{\mathbf{r}}_\mathbf{i} - \underline{\mathbf{r}}_\mathbf{cm}) = 0$

  becomes

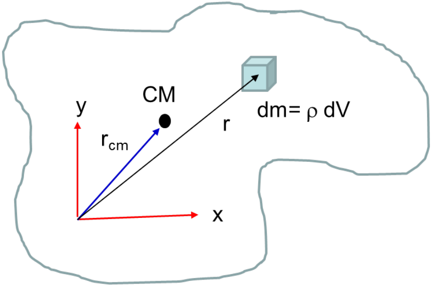
  $\int_V \rho(\underline{\mathbf{r}})(\underline{\mathbf{r}} - \underline{\mathbf{r}}_\mathbf{cm})dV = 0$

  where $\quad dm = \rho(\underline{\mathbf{r}})dV$

- Solve for r_{cm}

  $\underline{\mathbf{r}}_\mathbf{cm} = \frac{1}{M}\int_V \rho(\underline{\mathbf{r}})\ \underline{\mathbf{r}}\ dV$

  where M is the total mass in the volume



Example: the CM of Mount Ranier

Mount Ranier has approximately the shape of a cone (assume uniform density) and its height is 4400 m. At what height is the centre of mass?

We have cylindrical symmetry - just need to consider the y direction. Integrate from top ($y = 0$) to bottom ($y = h$)

- $y_{cm} = \frac{\int_0^h y dm}{M} = \frac{\int_0^h y dm}{\int_0^h dm}$

  $dm = \rho(\pi r^2)dy = \rho(\pi y^2 \tan^2\theta)dy$

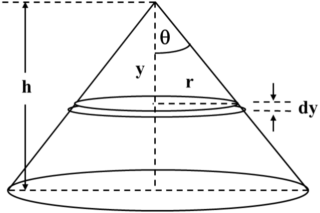
- $y_{cm} = \frac{\int_0^h y\rho(\pi y^2 \tan^2\theta)dy}{\int_0^h \rho(\pi y^2 \tan^2\theta)dy}$

  $y_{cm} = \frac{\int_0^h y^3 dy}{\int_0^h y^2 dy} = \frac{3h^4}{4h^3} = \frac{3h}{4}$

  (measured from the top)

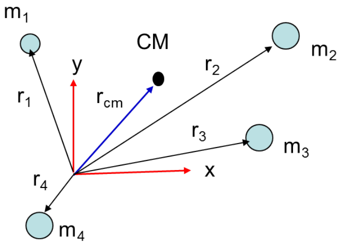
- $h = 4400m \rightarrow \; y_{cm} = 1100m$

  above the base



6.3.2 Velocity in the Centre of Mass frame

- The position of the centre of mass is given by:
  $\underline{\mathbf{r}}_{\mathbf{cm}} = \frac{1}{M}\sum_{i=1}^{n} m_i \, \underline{\mathbf{r}}_{\mathbf{i}}$
  where $M = \sum_{i=1}^{n} m_i$
- The velocity of the CM:
  $\underline{\mathbf{v}}_{\mathbf{cm}} = \underline{\dot{\mathbf{r}}}_{\mathbf{cm}} = \frac{1}{M}\sum_{i=1}^{n} m_i \, \underline{\dot{\mathbf{r}}}_{\mathbf{i}}$
  - In the Lab frame, total momentum $\underline{\mathbf{p}}_{tot}$:
    $\underline{\mathbf{p}}_{tot} = \sum_{i=1}^{n} m_i \, \underline{\dot{\mathbf{r}}}_{\mathbf{i}} = M\underline{\mathbf{v}}_{\mathbf{cm}}$



Hence the total momentum of a system in the Lab frame is equivalent to that of a single particle having a mass M and moving at a velocity $\underline{\mathbf{v}}_{\mathbf{cm}}$

6.3.3 Momentum in the CM frame

- Velocity of the CM: $\underline{\mathbf{v}}_{\mathbf{cm}} = \underline{\dot{\mathbf{r}}}_{\mathbf{cm}} = \frac{\sum_{i=1}^{n} m_i \ \underline{\dot{\mathbf{r}}}_{\mathbf{i}}}{\sum_i m_i} = \frac{\sum_{i=1}^{n} m_i \ \underline{\mathbf{v}}_{\mathbf{i}}}{\sum_i m_i}$
- Velocity of a body in the CM relative to Lab $\underline{\mathbf{v}}'_{\mathbf{i}} = \underline{\mathbf{v}}_{\mathbf{i}} - \underline{\mathbf{v}}_{\mathbf{cm}}$
- The total momentum of the system of particles in the CM:
- $\sum_i \underline{\mathbf{p}}'_{\mathbf{i}} = \sum_i m_i \underline{\mathbf{v}}'_{\mathbf{i}} = \sum_i m_i (\underline{\mathbf{v}}_{\mathbf{i}} - \underline{\mathbf{v}}_{\mathbf{cm}})$

$$= \sum_i m_i \underline{\mathbf{v}}_{\mathbf{i}} - \sum_i m_i \frac{\sum_j m_j \underline{\mathbf{v}}_{\mathbf{j}}}{\sum_j m_j} = \sum_i m_i \underline{\mathbf{v}}_{\mathbf{i}} - \sum_j m_j \underline{\mathbf{v}}_{\mathbf{j}} = 0$$

Hence the total momentum of a system of particles in the CM frame is equal to zero

- In addition, the total energy of the system is a minimum compared to all other inertial reference frames.

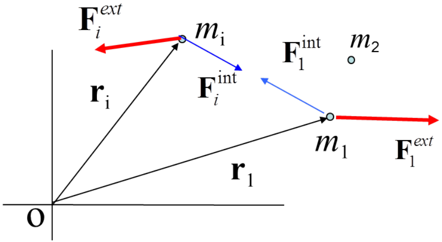
6.3.4 Motion of CM under external forces

- Force on particle i: $m_i \ddot{\underline{\mathbf{r}}}_i = \underline{\mathbf{F}}_{\mathbf{i}}{}^{ext} + \underline{\mathbf{F}}_{\mathbf{i}}{}^{int}$
- $$\underbrace{\sum_i^n m_i \ddot{\underline{\mathbf{r}}}_i}_{\text{all masses}} = \underbrace{\sum_i^n \underline{\mathbf{F}}_{\mathbf{i}}{}^{ext}}_{\text{external forces}} + \underbrace{\sum_i^n \underline{\mathbf{F}}_{\mathbf{i}}{}^{int}}_{\text{internal forces = zero}} = \sum_i^n \underline{\mathbf{F}}_{\mathbf{i}}{}^{ext}$$

- $\underline{\mathbf{r}}_{CM} = \sum_i \frac{m_i \underline{\mathbf{r}}_{\mathbf{i}}}{M}$

  where $M = \sum_i m_i$
- $\ddot{\underline{\mathbf{r}}}_{CM} = \sum_i \frac{m_i \ddot{\underline{\mathbf{r}}}_i}{M}$

  $\rightarrow M \ddot{\underline{\mathbf{r}}}_{CM} = \sum_i \underline{\mathbf{F}}_{\mathbf{i}}{}^{ext}$



The motion of the system is equivalent to that of a single particle having a mass M acted on by the sum of external forces

(The CM moves at constant velocity if no external forces)

6.4 Kinetic energy and the CM

- Lab kinetic energy : $T_{Lab} = \frac{1}{2}\sum m_i \underline{\mathbf{v}}_i^2$; $\underline{\mathbf{v}}_i' = \underline{\mathbf{v}}_i - \underline{\mathbf{v}}_{CM}$

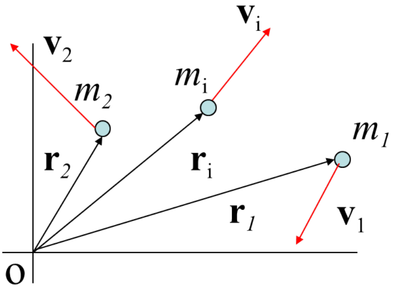
  where $\underline{\mathbf{v}}_i'$ is velocity of particle i in the CM

- $T_{Lab} = \frac{1}{2}\sum m_i \underline{\mathbf{v}}_i'^2 + \sum m_i \underline{\mathbf{v}}_i' \cdot \underline{\mathbf{v}}_{CM} + \frac{1}{2}\sum m_i \underline{\mathbf{v}}_{CM}^2$

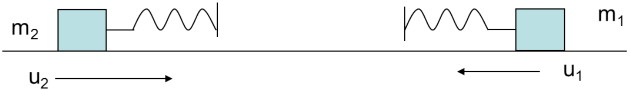
- But $\sum m_i \underline{\mathbf{v}}_i' \cdot \underline{\mathbf{v}}_{CM} = \underbrace{\frac{\sum m_i \underline{\mathbf{v}}_i'}{M}}_{=\,\underline{\mathbf{0}}} \cdot M\,\underline{\mathbf{v}}_{CM} = 0$

$\rightarrow$ $T_{Lab} = T_{CM} + \frac{1}{2}Mv_{cm}^2$

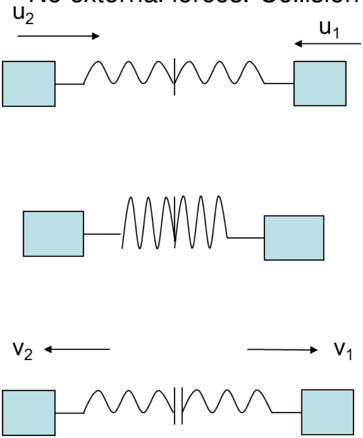
The kinetic energy in the Lab frame is equal to the kinetic energy in CM + the kinetic energy of the CM



7.1 Two-body collisions - general concepts



No external forces. Collision via massless springs or other force type.

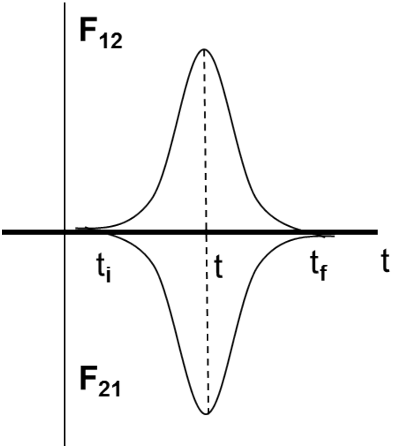


- t_i : collision starts. All energy is kinetic. $T_i = \frac{1}{2}m_1 u_1^2 + \frac{1}{2}m_2 u_2^2$
- t : collision peaks. Some kinetic is converted into potential (of the spring). $E = \frac{1}{2}m_1 \dot{x}_1{}^2 + \frac{1}{2}m_2 \dot{x}_2{}^2 + E_{int}$
- t_f : collision ends. All energy is kinetic again. $T_f = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2$; $T_i = T_f + \Delta E$ ($\leftarrow$ *inelastic*)

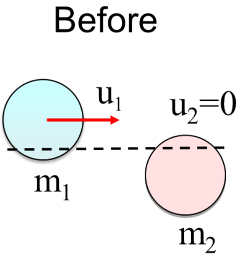
7.1.1 Momentum exchange and impulse

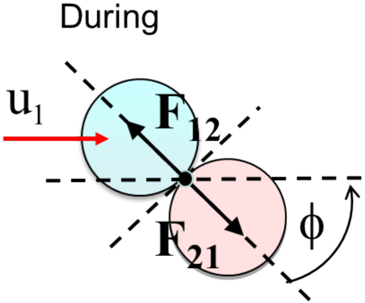
During collision: internal force causes change of momentum $\underline{\mathbf{F}} = \frac{d\underline{\mathbf{p}}}{dt}$

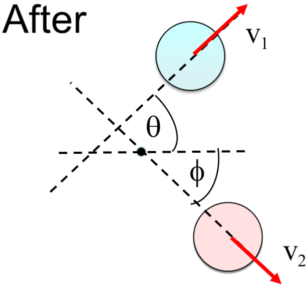
- At t_i : total momentum
  $\underline{\mathbf{p}} = \underline{\mathbf{p}}_1 + \underline{\mathbf{p}}_2 = m_1\underline{\mathbf{u}}_1 + m_2\underline{\mathbf{u}}_2$
- At t : $m_1 \;\rightarrow\; d\underline{\mathbf{p}}_1 = \underline{\mathbf{F}}_{12}dt$
  $m_2 \;\rightarrow\; d\underline{\mathbf{p}}_2 = \underline{\mathbf{F}}_{21}dt$
- At t_f : $m_1 \;\rightarrow\; \underline{\mathbf{v}}_1$ and $m_2 \;\rightarrow\; \underline{\mathbf{v}}_2$
- Impulse $\Delta\underline{\mathbf{p}}_1 = \underline{\mathbf{I}}_1 = \int_{t_i}^{t_f} \underline{\mathbf{F}}_{12}dt$
  $\Delta\underline{\mathbf{p}}_2 = \underline{\mathbf{I}}_2 = \int_{t_i}^{t_f} \underline{\mathbf{F}}_{21}dt$
- Since $\underline{\mathbf{F}}_{12} + \underline{\mathbf{F}}_{21} = 0$
  $\Delta\underline{\mathbf{p}}_1 + \Delta\underline{\mathbf{p}}_2 = 0$
- Momentum conserved.



7.1.2 An off-axis collision in 2D







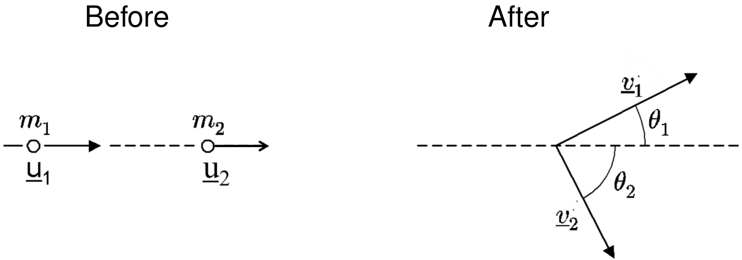
- Impulse is along line of centres

$$\Delta \underline{\mathbf{p}}_1 = \int_{t_i}^{t_f} \underline{\mathbf{F}}_{12} dt$$

$$\Delta \underline{\mathbf{p}}_2 = \int_{t_i}^{t_f} \underline{\mathbf{F}}_{21} dt$$

- $\underline{\mathbf{v}}_1 = \frac{1}{m_1} \int_{t_i}^{t_f} \underline{\mathbf{F}}_{12} dt + \underline{\mathbf{u}}_1$
- $\underline{\mathbf{v}}_2 = -\frac{1}{m_2} \int_{t_i}^{t_f} \underline{\mathbf{F}}_{12} dt$

7.2 Elastic collisions in the Lab frame

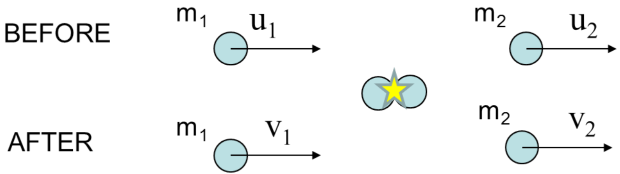


Conservation of momentum: $m_1\underline{\mathbf{u}}_1 + m_2\underline{\mathbf{u}}_2 = m_1\underline{\mathbf{v}}_1 + m_2\underline{\mathbf{v}}_2$

Conservation of energy: $\frac{1}{2}m_1\underline{\mathbf{u}}_1^2 + \frac{1}{2}m_2\underline{\mathbf{u}}_2^2 = \frac{1}{2}m_1\underline{\mathbf{v}}_1^2 + \frac{1}{2}m_2\underline{\mathbf{v}}_2^2$

[Note that the motion is in a plane, and the 2D representation can be trivially extended into 3D by rotation of the plane].

7.2.1 Elastic collisions in 1D in the Lab frame



- Momentum : $m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2 \quad (1)$

  $\rightarrow \quad m_1(v_1 - u_1) = m_2(u_2 - v_2) \quad (2)$

- Energy : $\frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2 = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$

  $\rightarrow \quad m_1(v_1 - u_1)(v_1 + u_1) = m_2(u_2 - v_2)(u_2 + v_2) \quad (3)$

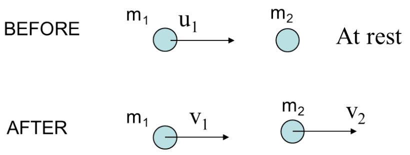
- Divide (2) & (3) :

  $\rightarrow \ (v_1 + u_1) = (u_2 + v_2) \ \rightarrow \ (u_1 - u_2) = (v_2 - v_1) \quad (4)$

  $\rightarrow$ Relative speed before collision = Relative speed after

7.2.2 Special case in 1D where target particle is at rest

- $u_2 = 0$; From (1) & (4) :
  $m_1 u_1 = m_1 v_1 + m_2(u_1 + v_1)$
- $v_1 = \frac{(m_1 - m_2)u_1}{m_1 + m_2}$
- Similarly :
  $m_1 u_1 = m_1(v_2 - u_1) + m_2 v_2$
- $v_2 = \frac{2m_1 u_1}{m_1 + m_2}$

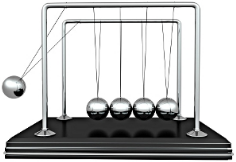


$$v_1 = \frac{m_1 - m_2}{m_1 + m_2} u_1 \quad \text{and} \quad v_2 = \frac{2m_1}{m_1 + m_2} u_1$$

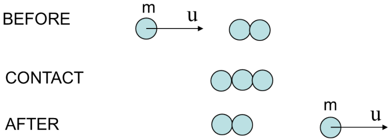
Special cases:

- $m_1 = m_2 : \rightarrow \quad v_1 = 0, \ v_2 = u_1$
  (complete transfer of momentum)
- $m_1 >> m_2$: Gives the limits $v_1 \rightarrow u_1, \ v_2 \rightarrow 2u_1$
  (m_2 has double u_1 velocity)
- $m_1 << m_2$: Gives the limits $v_1 \rightarrow -u_1, \ v_2 \rightarrow 0$
  ("brick wall" collision)

Example: Newton's cradle



Consider here just 3 balls



- If the balls are touching, the most general case is:

  Momentum after collision : $mu = mv_1 + mv_2 + mv_3$

  Energy after collision : $\frac{1}{2}mu^2 = \frac{1}{2}mv_1^2 + \frac{1}{2}mv_2^2 + \frac{1}{2}mv_3^2$

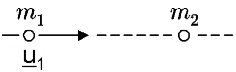
  2 equations, 3 unknowns
- The obvious solution: $v_1 = v_2 = 0, \quad v_3 = u$
- But other solution(s) possible:

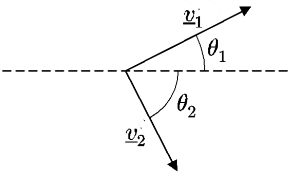
  Momentum : $mu = -\frac{1}{3}mu + \frac{2}{3}mu + \frac{2}{3}mu$

  Energy : $\frac{1}{2}mu^2 = \frac{1}{18}mu^2 + \frac{4}{18}mu^2 + \frac{4}{18}mu^2$
- So why does the simple solution always prevail?

7.2.3 Collision in 2D : equal masses, target at rest

$$m_1 = m_2 = m \,, \quad u_2 = 0$$





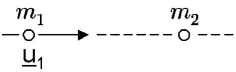
- Momentum: $m\underline{\mathbf{u}}_1 = m\underline{\mathbf{v}}_1 + m\underline{\mathbf{v}}_2 \;\rightarrow\; \underline{\mathbf{u}}_1 = \underline{\mathbf{v}}_1 + \underline{\mathbf{v}}_2$

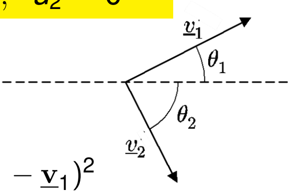
  Squaring $\rightarrow\; \underline{\mathbf{u}}_1^2 = \underline{\mathbf{v}}_1^2 + \underline{\mathbf{v}}_2^2 + 2\underline{\mathbf{v}}_1.\underline{\mathbf{v}}_2$
- Energy: $\frac{1}{2}m\underline{\mathbf{u}}_1^2 = \frac{1}{2}m\underline{\mathbf{v}}_1^2 + \frac{1}{2}m\underline{\mathbf{v}}_2^2 \;\rightarrow\; \underline{\mathbf{u}}_1^2 = \underline{\mathbf{v}}_1^2 + \underline{\mathbf{v}}_2^2$
- Hence $2\underline{\mathbf{v}}_1.\underline{\mathbf{v}}_2 = 0$

  $\rightarrow$ EITHER $\underline{\mathbf{v}}_1 = 0$ & $\underline{\mathbf{v}}_2 = \underline{\mathbf{u}}_1$ OR $(\theta_1 + \theta_2) = \frac{\pi}{2}$
- Either a head-on collision or opening angle is 90°

Relationship between speeds and angles

$$m_1 = m_2 = m \,, \quad u_2 = 0$$





- From cons. mom. $(\underline{\mathbf{v}}_2)^2 = (\underline{\mathbf{u}}_1 - \underline{\mathbf{v}}_1)^2$

  $\rightarrow \; v_2^2 = v_1^2 + u_1^2 - 2u_1 v_1 \cos\theta_1$
- Energy: $u_1^2 = v_1^2 + v_2^2 \;\rightarrow\; v_2^2 = u_1^2 - v_1^2$
- Equate : $2v_1^2 = 2u_1 v_1 \cos\theta_1$

$\cos\theta_1 = \frac{v_1}{u_1}$ and by symmetry $\cos\theta_2 = \frac{v_2}{u_1}$

- Note we can also do this via components of momentum :

  $\rightarrow \; u_1 = v_1 \cos\theta_1 + v_2 \cos\theta_2$ and $v_1 \sin\theta_1 = v_2 \sin\theta_2$

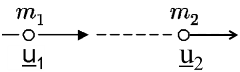
  $\rightarrow \; (u_1 - v_1 \cos\theta_1)^2 = v_2^2 \cos^2\theta_2$ and $v_1^2 \sin^2\theta_1 = v_2^2 \sin^2\theta_2$

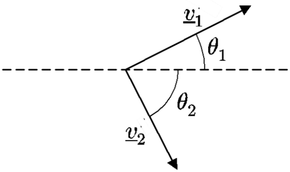
  $\rightarrow$ Add : $v_2^2 = v_1^2 \sin^2\theta_1 + u_1^2 - 2u_1 v_1 \cos\theta_1 + v_1^2 \cos^2\theta_1$

  $\rightarrow$ Gives : $v_2^2 = v_1^2 + u_1^2 - 2u_1 v_1 \cos\theta_1$

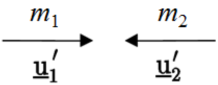
8.1 Elastic collisions in the CM frame

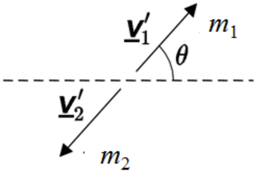
Lab frame:





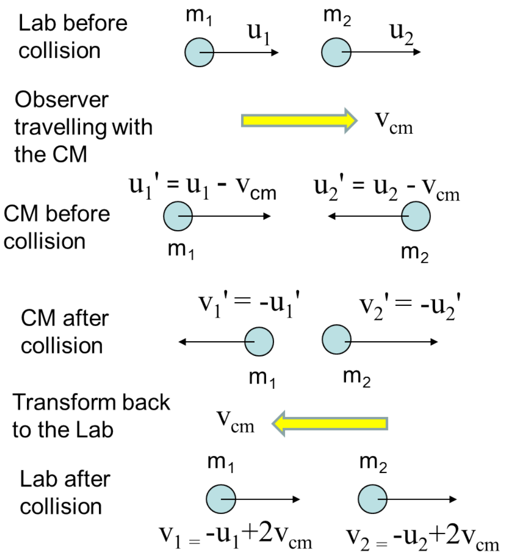
Centre of mass frame (zero momentum frame)





- Conservation of momentum in CM:
  $m_1\underline{\mathbf{u}}'_1 + m_2\underline{\mathbf{u}}'_2 = 0$; $m_1\underline{\mathbf{v}}'_1 + m_2\underline{\mathbf{v}}'_2 = 0$
- Conservation of energy in CM:
  $\frac{1}{2}m_1\underline{\mathbf{u}}'^2_1 + \frac{1}{2}m_2\underline{\mathbf{u}}'^2_2 = \frac{1}{2}m_1\underline{\mathbf{v}}'^2_1 + \frac{1}{2}m_2\underline{\mathbf{v}}'^2_2$

8.2 Lab to CM : 2-body 1D elastic collision



- $v_{cm} = \frac{(m_1 u_1 + m_2 u_2)}{(m_1 + m_2)}$
- Before in CM :
  $m_1 u_1' + m_2 u_2' = 0$
- After in CM :
  $m_1 v_1' + m_2 v_2' = 0$
- From last lecture
  $u_1' - u_2' = v_2' - v_1'$
- Sub for u_2', v_2' :
  $u_1'(1 + m_1/m_2) =$
  $-v_1'(1 + m_1/m_2)$
- $v_1' = -u_1'$
  $v_2' = -u_2'$

8.2.1 Collision in 1D : numerical example

$m_1 = 6\ kg$ | Lab frame | $m_2 = 4\ kg$

$u_1 = 3\ ms^{-1}$ | | $u_2 = 0.5 ms^{-1}$

1) Find CM velocity relative to laboratory frame :

$$v_{cm} = \frac{m_1 u_1 + m_2 u_2}{m_1 + m_2} = \frac{6\times3+0.5\times4}{10} = 2\,\text{ms}^{-1}$$

2) Transform initial velocities into CM : $u' = u - v_{cm}$

$$u'_1 = 3 - 2 = 1\,\text{ms}^{-1}\ ;\ u'_2 = 0.5 - 2 = -1.5\,\text{ms}^{-1}$$

$v_{cm} = 2\ ms^{-1}$

CM frame

$u'_1 = 1\ ms^{-1}$ | $u'_2 = -1.5\ ms^{-1}$

3) Conservation of energy : $v'_1 = -u'_1\ :\ v'_2 = -u'_2$

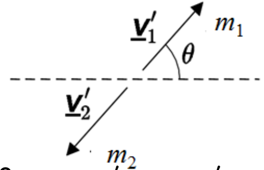
$v'_1 = -1\,\text{ms}^{-1}\ ;\ v'_2 = 1.5\,\text{ms}^{-1}$ after collision

4) Transform final velocities back to Laboratory frame :

$$v = v' + v_{cm}$$

$$v_1 = -1 + 2 = 1\,\text{ms}^{-1}\ ;\ v_2 = 1.5 + 2 = 3.5\,\text{ms}^{-1}$$

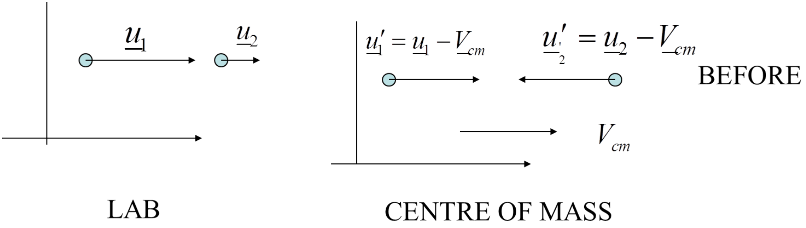
8.3 Relationship between speeds in CM in 2D



- Momentum : $m_1 \underline{\mathbf{u}}_1' + m_2 \underline{\mathbf{u}}_2' = 0$; $m_1 \underline{\mathbf{v}}_1' + m_2 \underline{\mathbf{v}}_2' = 0$ (1)
- Dot products : $m_1 u_1'^2 = -m_2 \underline{\mathbf{u}}_1' \cdot \underline{\mathbf{u}}_2'$; $m_2 u_2'^2 = -m_1 \underline{\mathbf{u}}_1' \cdot \underline{\mathbf{u}}_2'$
  $m_1 v_1'^2 = -m_2 \underline{\mathbf{v}}_1' \cdot \underline{\mathbf{v}}_2'$; $m_2 v_2'^2 = -m_1 \underline{\mathbf{v}}_1' \cdot \underline{\mathbf{v}}_2'$
- Energy : $\frac{1}{2} m_1 {u_1'}^2 + \frac{1}{2} m_2 {u'}_2^2 = \frac{1}{2} m_1 {v'}_1^2 + \frac{1}{2} m_2 {v'}_2^2$
  Hence $-(m_1 + m_2) \underline{\mathbf{u}}_1' \cdot \underline{\mathbf{u}}_2' = -(m_1 + m_2) \underline{\mathbf{v}}_1' \cdot \underline{\mathbf{v}}_2'$
- $\underline{\mathbf{u}}_1' \cdot \underline{\mathbf{u}}_2' = \underline{\mathbf{v}}_1' \cdot \underline{\mathbf{v}}_2'$: magnitudes $u_1' u_2' = v_1' v_2' \rightarrow$ back-to-back
- From (1), magnitudes $u_2' = -\frac{m_1}{m_2} u_1'$; $v_2' = -\frac{m_1}{m_2} v_1'$
- Hence $u_1'(-\frac{m_1}{m_2} u_1') = v_1'(-\frac{m_1}{m_2} v_1') \rightarrow u_1'^2 = v_1'^2$; $u_2'^2 = v_2'^2$
- $|v_1'| = |u_1'|$; $|v_2'| = |u_2'|$ $\rightarrow$

Speeds before = speeds after
Back-to-back in direction as
shown in diagrams.

8.4 Lab to CM : 2-body 2D elastic collision



1) Find centre of mass velocity v_{CM}

- $(\underline{\mathbf{u}}_1 - \underline{\mathbf{v}}_{CM})m_1 + (\underline{\mathbf{u}}_2 - \underline{\mathbf{v}}_{CM})m_2 = 0$
- $\rightarrow \underline{\mathbf{v}}_{CM} = \frac{m_1\underline{\mathbf{u}}_1 + m_2\underline{\mathbf{u}}_2}{m_1 + m_2}$

2) Transform initial Lab velocities to CM

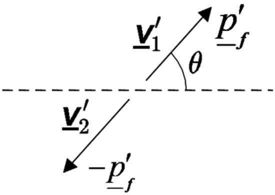
- $\underline{\mathbf{u}}'_1 = \underline{\mathbf{u}}_1 - \underline{\mathbf{v}}_{CM}$, $\underline{\mathbf{u}}'_2 = \underline{\mathbf{u}}_2 - \underline{\mathbf{v}}_{CM}$

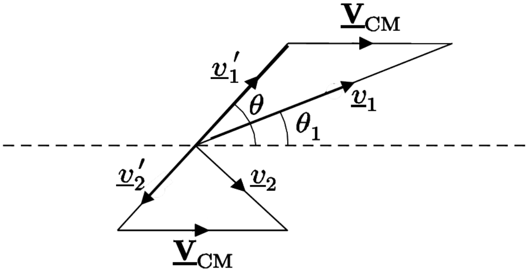
3) Get final CM velocities

- $|\underline{\mathbf{v}}'_1| = |\underline{\mathbf{u}}'_1|$; $|\underline{\mathbf{v}}'_2| = |\underline{\mathbf{u}}'_2|$

4) Transform vectors back to the Lab frame

- $\underline{\mathbf{v}}_1 = \underline{\mathbf{v}}_1' + \underline{\mathbf{v}}_{CM}$; $\underline{\mathbf{v}}_2 = \underline{\mathbf{v}}_2{}' + \underline{\mathbf{v}}_{CM}$

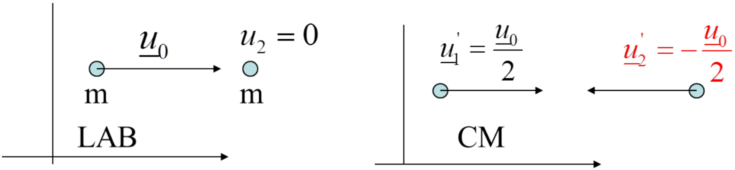




5) Can then use trigonometry to solve

9.1.1 Example 1: Equal masses, target at rest

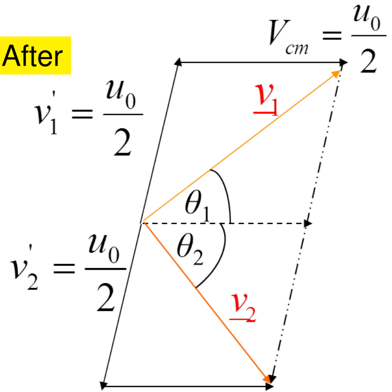
Before



Magnitude of velocities:

- $v_{CM} = \frac{m_1 u_1 + m_2 u_2}{m_1 + m_2} = \frac{u_0}{2}$
- $u_1' = u_0 - v_{CM} = \frac{u_0}{2}$
- $u_2' = -v_{CM} = -\frac{u_0}{2}$
- $|v_1'| = |u_1'| = \frac{u_0}{2}$
- $|v_2'| = |u_2'| = \frac{u_0}{2}$

After



Relationships between angles and speeds

Angles:

- Cosine rule:

  $$(\tfrac{u_0}{2})^2 = (\tfrac{u_0}{2})^2 + v_1^2 - 2v_1\tfrac{u_0}{2}\cos\theta_1$$

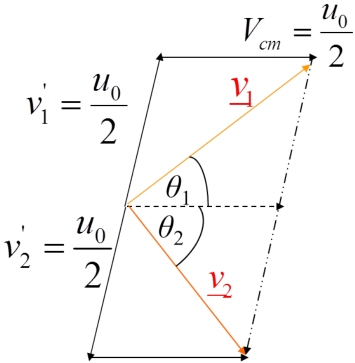
- $v_1 u_0 \cos\theta_1 = v_1^2$
- $\cos\theta_1 = \frac{v_1}{u_0}$ as before
- $\cos\theta_2 = \frac{v_2}{u_0}$

Opening angle:

- Cosine rule:

  $$u_0^2 = v_1^2 + v_2^2 - 2v_1 v_2 \cos(\theta_1 + \theta_2)$$

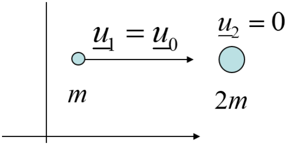
- But $u_0^2 = v_1^2 + v_2^2$ (conservation of energy)
- $\cos(\theta_1 + \theta_2) = 0 \quad \rightarrow \quad \theta_1 + \theta_2 = \frac{\pi}{2}$



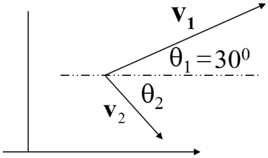
NB: Lines joining opposite corners of rhombus cross at 90°

9.1.2 Example 2: Elastic collision, $m_2 = 2m_1$, $\theta_1 = 30°$

Find the velocities v_1 and v_2 and the angle θ_2



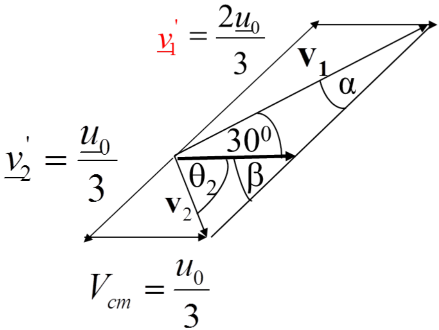
LAB before



LAB after

Magnitude of velocities:

- $v_{CM} = \frac{m_1 u_1 + m_2 u_2}{m_1 + m_2} = \frac{u_0}{3}$
- $u_1' = u_0 - v_{CM} = \frac{2u_0}{3}$
- $u_2' = -v_{CM} = -\frac{u_0}{3}$
- $|v_1'| = |u_1'| = \frac{2u_0}{3}$
- $|v_2'| = |u_2'| = \frac{u_0}{3}$



Relationships between angles and speeds

- Sine rule:

  $(\sin 30/\frac{2u_0}{3}) = (\sin\alpha/\frac{u_0}{3})$

  $\rightarrow \sin\alpha = \frac{1}{4} \rightarrow \alpha = 14.5°$
- $\beta = 30 + \alpha = 44.5°$
- $\sin 30/\frac{2u_0}{3} = \sin(180 - 44.5)/v_1$

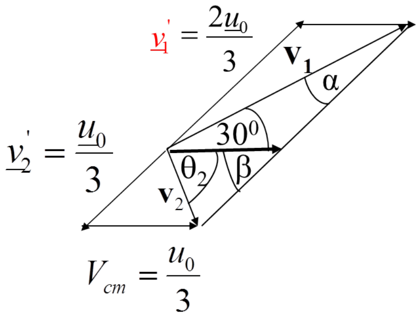
  $\rightarrow v_1 = 0.93u_0$
- Cosine rule:

  $v_2^2 = (\frac{u_0}{3})^2 + (\frac{u_0}{3})^2 - 2(\frac{u_0}{3})^2\cos\beta$

  $\rightarrow v_2 = 0.25u_0$
- Sine rule:

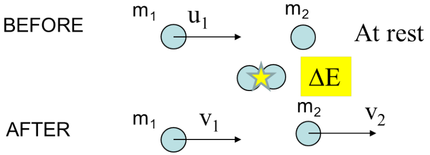
  $(\sin 44.5/v_2) = (\sin\theta_2/\frac{u_0}{3})$

  $\rightarrow \theta_2 = 68.0°$



9.2 Inelastic collisions in the Lab frame in 1D ($u_2 = 0$)

An *inelastic* collision is where energy is lost (or there is internal excitation).



- Take m_2 at rest & in 1D. Momentum : $m_1 u_1 = m_1 v_1 + m_2 v_2 \quad (1)$
- Energy : $\frac{1}{2} m_1 u_1^2 = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 + \Delta E \quad (2)$
- Square Equ.(1) and subtract $2m_1 \times$ Equ.(2)

$$\rightarrow \quad m_2(m_2 - m_1)v_2^2 + 2m_1 m_2 v_1 v_2 - 2m_1 \Delta E = 0$$

- Substitute for $m_1 v_1$ from Equ.1 to get quadratic in v_2

$$\rightarrow \quad m_2(m_2 + m_1)v_2^2 - 2m_1 m_2 u_1 v_2 + 2m_1 \Delta E = 0$$

- Solve, taking consistent solutions with elastic case ($\Delta E = 0$)

$$\rightarrow \quad v_2 = \frac{2m_1 m_2 u_1 + \sqrt{4m_1^2 m_2^2 u_1^2 - 8m_1 m_2 (m_1 + m_2)\Delta E}}{2m_2(m_1 + m_2)} \quad (3)$$

$$\rightarrow \quad v_1 = \frac{2m_1^2 u_1 - \sqrt{4m_1^2 m_2^2 u_1^2 - 8m_1 m_2 (m_1 + m_2)\Delta E}}{2m_1(m_1 + m_2)} \quad (4)$$

1D inelastic collisions viewed in the Lab frame ($u_2 = 0$)

- We see from Equ. (3) & (4) there is a *limiting case*: $4m_1^2 m_2^2 u_1^2 - 8m_1 m_2(m_1 + m_2)\Delta E \geq 0$
- i.e. $\Delta E \leq \frac{m_1 m_2 u_1^2}{2(m_1+m_2)}$
- This corresponds to the two bodies sticking together in a single object of mass $(m_1 + m_2) \quad \rightarrow \quad v_1 = v_2$
- From momentum cons. $\quad m_1 u_1 = m_1 v_1 + m_2 v_2$

  if $v_1 = v_2 = v$, then $v = \frac{m_1 u_1}{(m_1+m_2)}$ (the CM velocity)

For equal mass $m_1 = m_2$

$$v_2 \; , \; v_1 = \frac{u_1}{2}\left[1 \pm \sqrt{1 - \frac{4\Delta E}{mu_1^2}}\right]$$

9.2.1 Coefficient of restitution

General definition : $e = \frac{|\mathbf{v}_2 - \mathbf{v}_1|}{|\mathbf{u}_1 - \mathbf{u}_2|} = \frac{\text{Speed of relative separation}}{\text{Speed of relative approach}}$

- From Equ.(3) & (4) previously

$$v_2 - v_1 = \frac{2m_1 m_2 u_1 + \sqrt{4m_1^2 m_2^2 u_1^2 - 8m_1 m_2 (m_1+m_2)\Delta E}}{2m_2(m_1+m_2)} - \frac{2m_1^2 u_1 - \sqrt{4m_1^2 m_2^2 u_1^2 - 8m_1 m_2(m_1+m_2)\Delta E}}{2m_1(m_1+m_2)}$$

- Factorizing, then simplifying, then dividing by u_1 gives

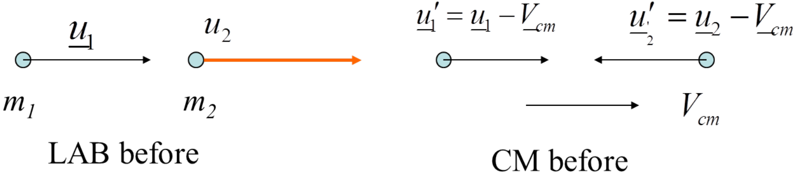
$$e = \sqrt{1 - \frac{2(m_1+m_2)\Delta E}{m_1 m_2 u_1^2}} = \sqrt{1 - \frac{\Delta E}{T'}}$$

where $T' = \frac{1}{2}\mu u_1^2$ with $\mu = \frac{m_1 m_2}{m_1+m_2}$ (the *reduced mass*)

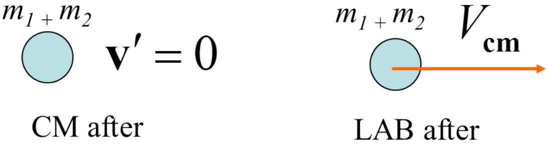
- We see later that T' is the initial energy in the CM frame, hence e is related to the fractional energy loss in this frame
- $e = 1$ completely elastic; $e = 0$ perfectly inelastic, in general $0 < e < 1$

9.3 Inelastic collisions viewed in the CM frame

Case of perfectly inelastic collision ($e = 0$)



After collision, total mass ($m_1 + m_2$) is at rest in CM:

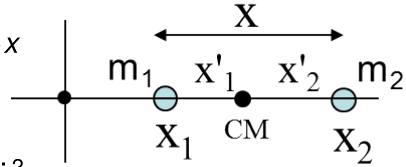


- KE in CM: $T_{CM} = T_{LAB} - \frac{1}{2}(m_1 + m_2)v_{CM}^2$
- Differentiate: Loss in KE $\Delta T_{CM} = \Delta T_{LAB}$ (obvious)
- Max. energy that can be lost = $T_{CM} =$
$= \frac{1}{2}m_1 u_1^2 + \frac{1}{2}m_2 u_2^2 - \frac{1}{2}(m_1 + m_2)v_{CM}^2$

9.3.1 Kinetic energy in the CM : alternative treatment

Revisit kinetic energy in the CM frame: $T_{Lab} = T_{CM} + \frac{1}{2} M v_{CM}^2$

- $T_{CM} = \frac{1}{2} m_1 u_1'^2 + \frac{1}{2} m_2 u_2'^2$
- $x_1' = -\frac{m_2}{m_1+m_2} x = -\frac{m_2}{M} x \; , \; x_2' = \frac{m_1}{M} x$
- $u_1' = -\frac{m_2}{M} \dot{x} \quad , \quad u_2' = \frac{m_1}{M} \dot{x}$
- $T_{CM} = \frac{1}{2} \left(m_1 (-\frac{m_2}{M})^2 + m_2 (\frac{m_1}{M})^2 \right) \dot{x}^2$
- $T_{CM} = \frac{1}{2} \frac{m_1 m_2}{M^2} (m_2 + m_1) \dot{x}^2 = \frac{1}{2} \frac{m_1 m_2}{M} \dot{x}^2$
- Also $\dot{x} = \dot{x}_2 - \dot{x}_1 = u_2' - u_1'$



$$T_{CM} = \frac{1}{2} \frac{m_1 m_2}{M} \dot{x}^2 = \frac{1}{2} \mu \dot{x}^2 = \frac{1}{2} \mu (u_1' - u_2')^2 = \frac{1}{2} \mu (u_1 - u_2)^2$$

These expressions give the CM kinetic energy in terms of the relative velocities in the CM & Lab and the reduced mass μ

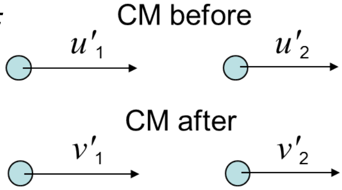
9.3.2 Coefficient of restitution in the CM

- Initial KE in the CM : $T^i_{CM} = \frac{1}{2}\mu(u'_1 - u'_2)^2$
- Final KE in the CM : $T^f_{CM} = \frac{1}{2}\mu(v'_1 - v'_2)^2$
- Conservation of energy : $T^i_{CM} = T^f_{CM} + \Delta E$

$$\rightarrow \quad \frac{1}{2}\mu(u'_1 - u'_2)^2 = \frac{1}{2}\mu(v'_1 - v'_2)^2 + \Delta E$$

$$\rightarrow \quad \left(\frac{v'_1 - v'_2}{u'_1 - u'_2}\right)^2 = 1 - \frac{\Delta E}{T^i_{CM}}$$

$$\rightarrow \quad \left(\frac{v'_1 - v'_2}{u'_1 - u'_2}\right) = \pm\sqrt{1 - \frac{\Delta E}{T^i_{CM}}}$$



Same expression as before with $T' = T^i_{CM}$

Coefficient of restitution

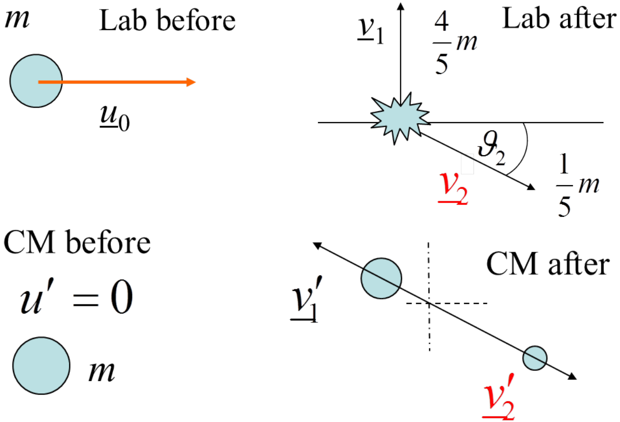
$$e = \frac{|\mathbf{v}'_2 - \mathbf{v}'_1|}{|\mathbf{u}'_1 - \mathbf{u}'_2|}_{CM} = \frac{|\mathbf{v}_2 - \mathbf{v}_1|}{|\mathbf{u}_1 - \mathbf{u}_2|}_{LAB} = \sqrt{1 - \frac{\Delta E}{T^i_{CM}}}$$

ONLY in CM frame can ALL the KE be used to create ΔE

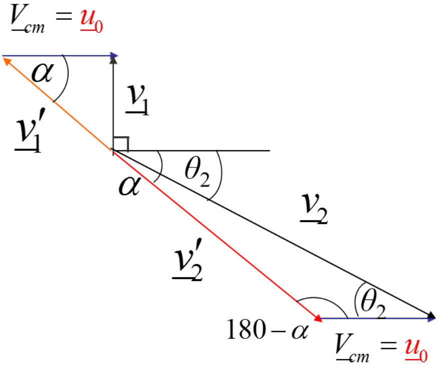
$\Rightarrow$ For $e = 0$ the two particles coalesce and are at rest in CM

9.3.3 Example of inelastic process

A calcium nucleus (A=20), mass m, travels with velocity u_0 in the Lab. It decays into a sulphur nucleus (A=16), mass $\frac{4}{5}m$, and an α-particle (A=4), mass $\frac{1}{5}m$. Energy ΔT is released as KE in the calcium rest frame (CM). A counter in the Lab detects the sulphur nucleus at 90° to the line of travel. What is the speed and angle of the α-particle in the Lab?



- Energy ΔT is released as KE in the CM. $v_{CM} = u_0$
- Momentum in CM:
  $\frac{4}{5} m v_1' - \frac{1}{5} m v_2' = 0$
  $\rightarrow \quad v_2' = 4 v_1'$
- Energy: $\Delta T = \frac{1}{2}(\frac{4}{5}m){v_1'}^2 + \frac{1}{2}(\frac{1}{5}m) 16 {v_1'}^2 = 2m{v_1'}^2$
  $\rightarrow \quad v_1' = [\frac{\Delta T}{2m}]^{\frac{1}{2}}$
  $\rightarrow \quad v_2' = [\frac{8\Delta T}{m}]^{\frac{1}{2}}$
- Transform to Lab by boosting by $v_{CM} (= u_0)$



- $\cos\alpha = \frac{u_0}{v_1'} = [\frac{2mu_0^2}{\Delta T}]^{\frac{1}{2}}$
- Cosine rule: $v_2^2 = {v_2'}^2 + {u_0}^2 + 2 v_2' u_0 \cos\alpha$
- Sine rule: $\frac{\sin\theta_2}{v_2'} = \frac{\sin\alpha}{v_2}$

Solve for v_2 , θ_2

10.1 Resisted motion and limiting speed

- Newton II: $m\frac{dv}{dt} = F_{ext} + F_R$ where F_{ext} is the external force and F_R is a resistive force
- If $F_{ext} = 0$ and $F_R \propto$ velocity, then $v \propto \exp(-\alpha t)$ (see Lecture 4)
- If $F_{ext} \neq 0$ and e.g. $F_R \propto -v^n$ then there exists a *limiting speed* corresponding to $\frac{dv}{dt} = 0$ that satisfies $F_R = -F_{ext}$

10.2 Air resistance

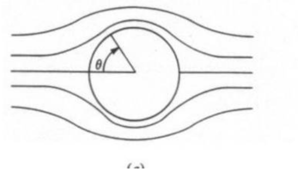
$$F_R = \underbrace{av}_{\text{Laminar flow}} + \underbrace{bv^2}_{\text{Turbulent flow}}$$

- Laminer flow : Stoke's Law
  $F = 6\pi\eta r v$

  r is the radius of the sphere

  v is the velocity of the sphere

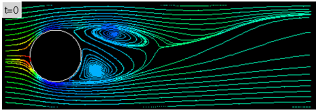
  η is the viscosity of the fluid



- Turbulent flow : S $F = \frac{1}{2}\pi\rho C_d r^2 v^2$

  ρ is the density of the fluid

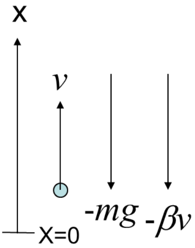
  C_d is the drag coefficient (e.g. for a smooth sphere $C_d \sim 0.47$



10.3 Example 1 : Resistive force, $F_R \propto v$

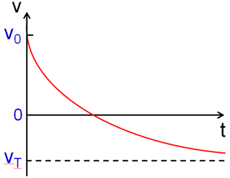
- Body fired *vertically upwards* under gravity $\rightarrow$ air resistance $\propto$ velocity

  $\rightarrow \; v = v_0$ & $x = 0$ at $t = 0$



- Equation of motion: $m\frac{dv}{dt} = -mg - \beta v$
  - $\int_{v_0}^{v} \frac{dv}{g+\alpha v} = -\int_0^t dt$ where $\alpha = \frac{\beta}{m}$
  - $\left[\frac{1}{\alpha} log_e(g + \alpha v)\right]_{v_0}^{v} = [-t\,]_0^t$
  - $log_e\left(\frac{(g+\alpha v)}{(g+\alpha v_0)}\right) = -\alpha t \;\rightarrow\; 1+\frac{\alpha v}{g} = \left(1 + \frac{\alpha v_0}{g}\right)\exp(-\alpha t)$

$$v = \frac{g}{\alpha}\left[\left(1 + \frac{\alpha v_0}{g}\right)\exp(-\alpha t) - 1\right]$$



- Terminal (limiting) velocity: $t \rightarrow \infty$, $v_T \rightarrow -\frac{g}{\alpha}$
- Can show by expansion, as $\alpha \rightarrow 0$, $v \rightarrow v_0 - gt$

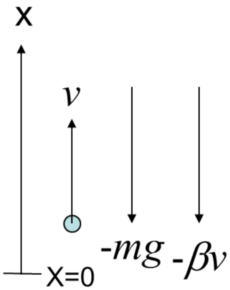
Maximum height and distance travelled for $F_R \propto v$

- $v = \frac{g}{\alpha}\left[(1+\frac{\alpha v_0}{g})\exp(-\alpha t) - 1\right]$
- At maximum height $\rightarrow v = 0, t = t_{max}$

$$\rightarrow \exp(-\alpha t_{max}) = (1+\frac{\alpha v_0}{g})^{-1}$$

$$t_{max} = \frac{1}{\alpha}\log_e\left(1+\frac{\alpha v_0}{g}\right)$$

Can expand log to show :
$t_{max} \rightarrow \frac{v_0}{g}$ when $\alpha \rightarrow 0$



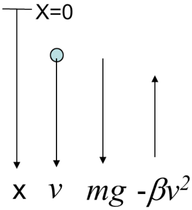
- Distance travelled :
- $x = \int_0^t \frac{g}{\alpha}\left[(1+\frac{\alpha v_0}{g})\exp(-\alpha t) - 1\right] dt$

$$= \frac{g}{\alpha}\left[-\frac{1}{\alpha}(1+\frac{\alpha v_0}{g})\exp(-\alpha t) - t\right]_0^t$$

$$x = \frac{g}{\alpha}\left[\left(\frac{1}{\alpha}(1+\frac{\alpha v_0}{g})(1-\exp(-\alpha t)\right) - t\right]$$

Can show by expansion
$x \rightarrow v_0 t - \frac{1}{2}gt^2$
when $\alpha \rightarrow 0$

10.4 Example 2: Resistive force, $F_R \propto v^2$



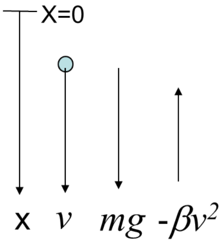
- Body falls *vertically downwards* under gravity with air resistance $\propto$ [velocity]^2, $v = 0$, $x = 0$ at $t = 0$
- Equation of motion: $m\frac{dv}{dt} = mg - \beta v^2$
- Terminal velocity when $\frac{dv}{dt} = 0$: $v_T = \sqrt{\frac{mg}{\beta}}$
  - Equation of motion becomes $\frac{dv}{dt} = g\left(1 - v^2/v_T^2\right)$
  - Integrate $\int_0^v \frac{dv}{g\left(1 - v^2/v_T^2\right)} = \int_0^t dt$
  - Standard integral : $\int \frac{1}{1-z^2}\, dz = \frac{1}{2}\log_e\left(\frac{1+z}{1-z}\right)$
  - $\left[\frac{v_T}{2g} log_e\left(\frac{1+v/v_T}{1-v/v_T}\right)\right]_0^v = t \;\rightarrow\; \frac{1+v/v_T}{1-v/v_T} = \exp(t/\tau)$, where $\tau = \frac{v_T}{2g}$

    $\rightarrow\; (1 - \frac{v}{v_T}) = (1 + \frac{v}{v_T})\exp(-\frac{t}{\tau})$

Velocity as a function of time:

$$v = v_T\left[\frac{1-\exp(-t/\tau)}{1+\exp(-t/\tau)}\right]$$

Velocity as a function of distance for $F_R \propto v^2$



- Equation of motion: $\frac{dv}{dt} = g\left(1 - v^2/v_T^2\right)$
- Write $\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$
  - $\int_0^v \frac{v\,dv}{g\left(1 - v^2/v_T^2\right)} = \int_0^x dx$
  - $\left[-\frac{v_T^2}{2g} log_e\left(1 - v^2/v_T^2\right)\right]_0^v = x$

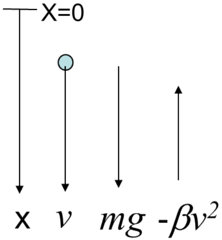
$$\rightarrow \quad \left(1 - v^2/v_T^2\right) = \exp\left(-x/x_T\right) \quad , \quad \text{where } x_T = \frac{v_T^2}{2g}$$

$$v^2 = v_T^2\left[1 - \exp\left(-x/x_T\right)\right]$$

To get x vs. t integrate again : $\rightarrow \int_0^t dt = \int_0^x \frac{dx}{v}$

10.4.1 Work done on the body by the force for $F_R \propto v^2$

- Equation of motion: $m\frac{dv}{dt} = mg - \beta v^2$
- Work done:
$$\int F dx = \underbrace{\int_o^x mg\, dx}_{Conservative} - \underbrace{\int_0^x \beta v^2 dx}_{Dissipative}$$
- Conservative term : Work done $= mgx$
- Dissipative term : Work done
$$= -\int_0^x \beta v^2 dx = -\int_0^x \beta v_T^2 \left[1 - \exp(-x/x_T)\right] dx$$
$$= -\beta v_T^2[x + \underbrace{x_T\left(\exp(-x/x_T) - 1\right)}_{=-x_T v^2/v_T^2}]$$
$$= -\beta v_T^2(x - v^2/2g) = -mg[x - v^2/2g]$$



$$v_T^2 = \frac{mg}{\beta}$$
$$x_T = \frac{v_T^2}{2g}$$

Energy dissipated = $\frac{1}{2}mv^2 - mgx$ As expected.