LECTURES 11 - 19

INTRODUCTION TO CLASSICAL MECHANICS

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HT 2017

OUTLINE : INTRODUCTION TO MECHANICS LECTURES 11-19

19.3.1 Example: binary star

Programme for Hilary term (20 lectures)

- Lectures 1-5

  Rocket motion. Motion in B and E fields
- Lectures 6-10

  Central forces (orbits)
- Lectures 11-15

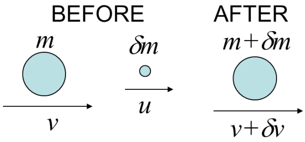
  Rotational dynamics (rigid body etc)
- Lectures 16-20

  Lagrangian dynamics

- Plus 4 problem sets for your enjoyment

11.1 Variable mass : a body acquiring mass

- A body of mass m has velocity v. In time δt it it acquires mass δm, which is moving along v direction with velocity u

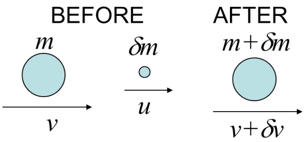


- The change in mass m is $m + \delta m$, the change in velocity v is $v + \delta v$
  - Case 1: No external force. Change of momentum Δp

$$\Delta p = \underbrace{(m + \delta m)(v + \delta v)}_{After} - \underbrace{(mv + u\delta m)}_{Before} = 0$$

  - $mv + m\delta v + v\delta m + \underbrace{\delta m \delta v}_{Ignore} - mv - u\delta m = m\delta v + (v - u)\,\delta m = 0$
  - Divide by δt (time over which mass acquisition occurs) :
$\frac{\Delta p}{\delta t} = m\frac{\delta v}{\delta t} + \underbrace{(v - u)}_{Relative\ velocity,\, w} \frac{\delta m}{\delta t} = 0$
  - As $\delta t \to 0$, $m\frac{dv}{dt} + w\frac{dm}{dt} = 0$ (in this case $\frac{dv}{dt}$ is -ve as expected)

A body acquiring mass - with external force



- Case 2: Application of an external force F
- NII : change of momentum $= \Delta p = F\delta t = m\delta v + w\,\delta m$ as before, where $w = (v - u)$
- Divide by δt and let $\delta t \to 0$

$$m\frac{dv}{dt} + w\frac{dm}{dt} = F$$

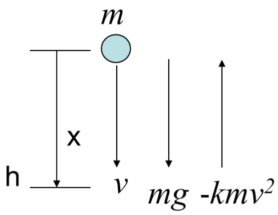
Note : ONLY in the case when $u = 0$ does $\frac{d}{dt}(mv) = F$

11.2 Example - the raindrop

An idealised raindrop has initial mass m_0, is at height h above ground and has zero initial velocity. As it falls it acquires water (added from rest) such that its increase in mass at speed v is given by $dm/dt = bmv$ where b is a constant. The air resistance is of the form kmv^2 where k is a constant.

- Formulate the equation of motion :
- $m\frac{dv}{dt} + w\frac{dm}{dt} = F$

  $\rightarrow \quad m\frac{dv}{dt} + w\frac{dm}{dt} = mg - kmv^2$
- $w = v$ (since $u = 0$) ; $\frac{dm}{dt} = bmv$
- $\frac{dv}{dt} + (b+k)v^2 = g$



  - Terminal velocity :
  - $\frac{dv}{dt} = 0 \quad \rightarrow \quad v_T = \sqrt{\frac{g}{b+k}}$

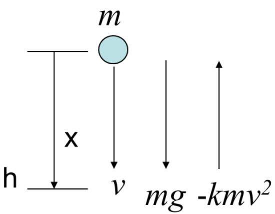
The raindrop, continued

- Calculate raindrop mass vs. distance
- $\frac{dm}{dt} = bmv$
- $\frac{dm}{dx} = \frac{dm}{dt}\frac{dt}{dx} = \frac{bmv}{v} = bm$

  $\rightarrow \quad \frac{dm}{m} = bdx$

  Integrate : $\quad [\log_e m]_{m_0}^{m} = [bx]_0^x$
- $m = m_0 \exp(bx)$

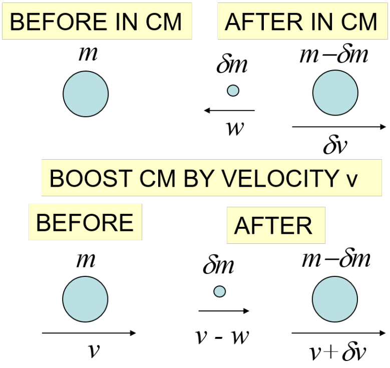
  (Mass grows exponentially with x)



  - What is its speed at ground level ?
  - $\frac{dv}{dt} + (b+k)v^2 = g \quad \rightarrow \quad \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$
  - $\int_0^{v_h} \frac{vdv}{g-(b+k)v^2} = \int_0^h dx \quad \rightarrow \quad h = \left[-\left(\frac{\log_e(g-(b+k)v^2)}{2(b+k)}\right)\right]_0^{v_h}$
  - Solving : $\quad v_h = \sqrt{\frac{g}{b+k}\left[1 - \exp\left(-2h(b+k)\right)\right]}$

11.3 Ejecting mass : the rocket equation

- A body of mass m has velocity v. In time δt it *ejects* mass δm, with relative velocity w to the body



- Change of momentum $\Delta p = \underbrace{\delta m(v - w) + (m - \delta m)(v + \delta v)}_{\textit{After}} - \underbrace{mv}_{\textit{Before}}$

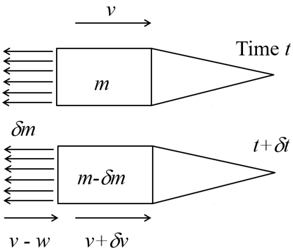
$$= v\delta m - w\delta m + mv + m\delta v - v\delta m - \underbrace{\delta m \delta v}_{\textit{Ignore}} - mv$$

- With external force : $F = \frac{\Delta p}{\delta t} = m\frac{\delta v}{\delta t} - w\frac{\delta m}{\delta t}$
- As $\delta t \to 0$, $\frac{\delta v}{\delta t} \to \frac{dv}{dt}$ & $\frac{\delta m}{\delta t} \to -\frac{dm}{dt}$ (as $\frac{\delta m}{\delta t}$ is +ve but $\frac{dm}{dt}$ is -ve)
- Hence, again, $F = m\frac{dv}{dt} + w\frac{dm}{dt}$ the rocket equation

11.4 The rocket : horizontal launch

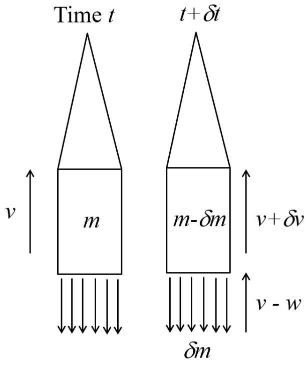
- Rocket equation:
  $m\frac{dv}{dt} + w\frac{dm}{dt} = F = 0$ (no gravity)
- Assume mass is ejected with *constant* relative velocity to the rocket w
- $m\,dv = -w\,dm \quad \rightarrow \quad dv = -w\frac{dm}{m}$
- Initial/final velocity = v_i, v_f
  Initial/final mass = m_i, m_f
- $\int_{v_i}^{v_f} dv = -w \int_{m_i}^{m_f} \frac{dm}{m}$
- $v_f - v_i = w \log_e (m_i / m_f)$

  This expression gives the dependence of rocket velocity as a function of its mass



12.1 The rocket : vertical launch

- Rocket equation:
  $m\frac{dv}{dt} + w\frac{dm}{dt} = F$
- Rocket rises against gravity
  $F = -mg$
- Mass is ejected at *constant* velocity w relative to the rocket
- Rocket ejects mass uniformly:
  $m = m_0 - \alpha t$
  $\rightarrow \frac{dm}{dt} = -\alpha$



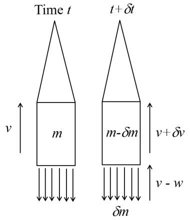
- Now consider upward motion:
- $mdv = (-mg + w\alpha)dt \quad \rightarrow \quad \int_{v_i}^{v_f} dv = \int_{t_i}^{t_f} \left(-g + \frac{w\alpha}{m_0 - \alpha t}\right) dt$
- $v_f - v_i = \left[-g(t_f - t_i) - w \log_e \frac{(m_0 - \alpha t_f)}{(m_0 - \alpha t_i)}\right]$
  $= [-g(t_f - t_i) - w \log_e(m_f / m_i)]$

Rocket vertical launch, continued

The rocket starts from rest at $t = 0$; half the mass is fuel. What is the velocity and height reached by the rocket at burn-out at time $t = T$?

- $v = \left[-gt - w\log_e \frac{(m_0 - \alpha t)}{(m_0)}\right] = \left[-gt - w\log_e\left(1 - \frac{\alpha}{m_0}t\right)\right] = \frac{dx}{dt}$
- What is the condition for the rocket to rise ? $\rightarrow \frac{dv}{dt} > 0$

  At $t = 0,\ m = m_0,\ \frac{dm}{dt} = -\alpha \ :\ \alpha w - m_0 g > 0 \ \rightarrow\ w > \frac{m_0 g}{\alpha}$



- $m = m_0 - \alpha t$; at burnout $t = T$, $m = \frac{m_0}{2} \rightarrow \alpha = \frac{m_0}{2T}$
- Maximum velocity is at the burn-out of the fuel:

  At $t = T$: $v_{max} = -gT + w\log_e 2$

  Height : $\int_0^x dx = \int_0^T \left[-gt - w\log_e\left(1 - \frac{\alpha}{m_0}t\right)\right] dt$

  - Standard integral : $\int \log_e z\, dz = z\log_e z - z$
  - $x = -\frac{gT^2}{2} + \frac{w\, m_o}{\alpha}\left[\left(1 - \frac{\alpha}{m_0}t\right)\left(\log_e\left(1 - \frac{\alpha}{m_0}t\right)\right) - \left(1 - \frac{\alpha}{m_0}t\right)\right]_0^T$
  - After simplification : $x = -\frac{gT^2}{2} + wT(1 - \log_e 2)$

12.2 The 1-stage vs. 2-stage rocket

A two stage rocket is launched vertically from earth, total mass $M_0 = 10000$ kg and carries an additional payload of $m = 100$ kg. The fuel is 75% of the mass in both stages; burn rate $\alpha = 500$ kg s^{-1}, thrust velocity $w = 2.5$ km s^{-1}. The mass of the 2nd stage is 900 kg.

i) Calculate the final speed for the equivalent single stage rocket

ii) Find the final speed of the 2-stage rocket

(i) Single stage rocket :

- Time to burn-out :

$$\left|\frac{\Delta m}{\Delta t}\right| = \alpha \rightarrow T = 0.75(M_1 + M_2)/500 \text{ [kg s}^{-1}]$$

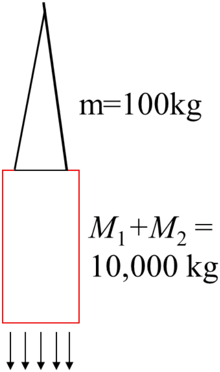
- From before : $v_f = -gT + w\log_e(m_i/m_f)$

$$m_i = M_1 + M_2 + m \;\; ; \;\; m_f = 0.25(M_1 + M_2) + m$$

- Single stage : $v_f = 3.25\,\text{km s}^{-1}$
- Earth's escape speed :

$$\frac{1}{2}mv^2 > \frac{GM_e m}{R_E} \rightarrow v_{esc} = \sqrt{\frac{2GM_E}{R_E}}$$

$$\rightarrow v_{esc} = 11.2\,\text{km s}^{-1} \;\; (i.e.\; v_f < v_{esc})$$



(ii) The 2-stage rocket

Stage 1

- $|\frac{\Delta m}{\Delta t}| = \alpha \quad \rightarrow T = 0.75M_1/500$ [kg s^{-1}]
- $v_1 = -gT + w\log_e(m_i/m_f)$

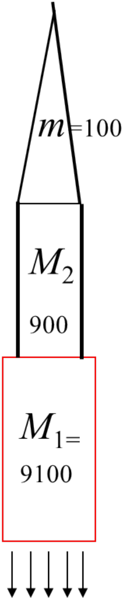
  $m_i = M_1 + M_2 + m \; ; \; m_f = 0.25M_1 + M_2 + m$
- After first stage :$v_f = 2.68$ km s^{-1}

Stage 2

- $|\frac{\Delta m}{\Delta t}| = \alpha \quad \rightarrow T = 0.75M_2/500$ [kg s^{-1}]
- $v_2 = v_1 - gT + w\log_e(m_i/m_f)$

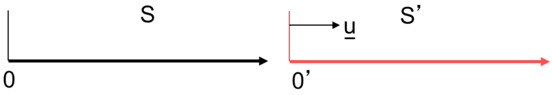
  $m_i = M_2 + m \; ; \; m_f = 0.25M_2 + m$
- After second stage :

  $v_2 = 2.68 + 2.80 = 5.48$ km s^{-1}
- Need a 3rd stage, more thrust or less payload to escape



12.3 Non-inertial reference frames

A frame in which Newton's first law is not satisfied - the frame is accelerating (i.e. subject to an external force)

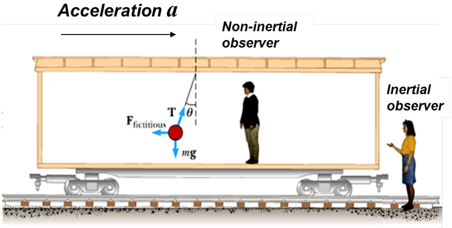


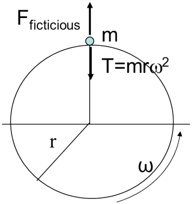
Recall Galilean transformation but now $\underline{\mathbf{u}}$ varies with time :

	Galilean transformation	Accelerating frame
Position	$\underline{\mathbf{r}}' = \underline{\mathbf{r}} - \underline{\mathbf{u}}t$	$\underline{\mathbf{r}}' = \underline{\mathbf{r}} - \int \underline{\mathbf{u}}(t)\, dt$
Velocity	$\underline{\mathbf{v}}' = \underline{\mathbf{v}} - \underline{\mathbf{u}}$	$\underline{\mathbf{v}}' = \underline{\mathbf{v}} - \underline{\mathbf{u}}(t)$
Acceleration	$\frac{d\underline{\mathbf{v}}'}{dt} = \frac{d\underline{\mathbf{v}}}{dt}$	$\frac{d\underline{\mathbf{v}}'}{dt} = \frac{d\underline{\mathbf{v}}}{dt} - \frac{d\underline{\mathbf{u}}}{dt}$
Force on mass	$\underline{\mathbf{F}}'(\underline{\mathbf{r}}') = F(\underline{\mathbf{r}}) = m\frac{d\underline{\mathbf{v}}}{dt}$	$\underline{\mathbf{F}}'(\underline{\mathbf{r}}') = \underline{\mathbf{F}}(\underline{\mathbf{r}}) - m\frac{d\underline{\mathbf{u}}}{dt}$

- So even if $F(\underline{\mathbf{r}}) = 0$, from NII, there is an *apparent (or "ficticious") force* acting in S' of $\underline{\mathbf{F}}'(\underline{\mathbf{r}}') = -m\frac{d\underline{\mathbf{u}}}{dt}$

12.3.1 Commonplace examples





Accelerating train

- In the inertial frame

  $\sum F_x = T \sin\theta = ma$

  $\sum F_y = T \cos\theta - mg = 0$

- In the non-inertial frame

  $\sum F_x' = T \sin\theta - F_{fictitious}$

  $\sum F_y' = T \cos\theta - mg = 0$

In NIF need to introduce $F_{fictitious} = ma$ to explain the displacement of the bob.

Mass rotating in a circle

- In the inertial frame

  Centripetal acceleration provided by tension in the string

  $T = mr\omega^2$

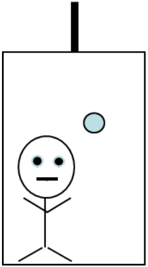
- In the non-inertial frame

  The block is at rest and its acceleration is zero

In NIF need $F_{fictitious} = mr\omega^2$ (centrifugal force) to balance the tension in the string.

12.3.2 Example : accelerating lift

- First consider the lift in free-fall
- The ball is “weightless” (stationary or moves at constant velocity) according to an observer in the lift $\rightarrow$ lift becomes an inertial frame (like in deep space) : NI.
- To this observer the fictitious acceleration $\left(-\frac{d\mathbf{u}}{dt}\right)$ balances the gravitational acceleration



Observer in accelerating lift

- Lift plus passenger (total mass M) is now accelerated upwards with force F. Passenger drops ball mass m' from height h $(M >> m')$.
- Total force on lift $F_{tot} = F - Mg = Ma$

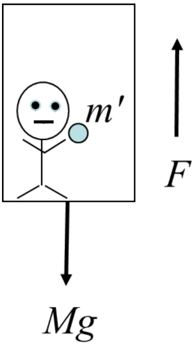
  Acceleration of lift $a = \frac{F}{M} - g$
- According to passenger in lift frame, *downwards* acceleration of ball is

  $$= (\frac{F}{M} - g) + g \;= \frac{F}{M} \text{ (downwards)}$$
- Hence weight of ball appears to passenger to be $(F/M) \times m'$
- Check this:

  If $F = 0$, free-fall, ball is weightless

  If $F = Mg$, lift is stationary, ball has weight $m'g$

  - Time for ball to reach floor, use $h = \frac{1}{2}at^2 \;\rightarrow\; t = \sqrt{\frac{2hM}{F}}$

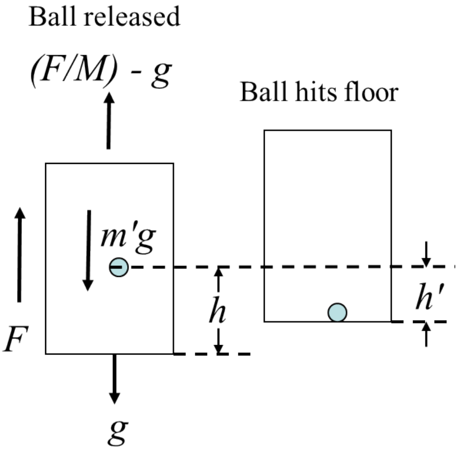


Observer watching from a frame outside the lift

- Observer watches ball fall with acceleration g and the lift rise with acceleration $\frac{F}{M} - g$
- Equate times when ball reaches floor:

$$t = \underbrace{\sqrt{\frac{2h'}{g}}}_{\text{ball falling}} = \underbrace{\sqrt{\frac{2(h-h')}{F/M - g}}}_{\text{lift rising}}$$

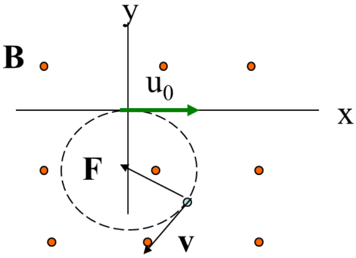
- Solve for h' $\rightarrow$ $h' = \frac{Mgh}{F}$
- Substitute back $\rightarrow$ $t = \sqrt{\frac{2Mh}{F}}$ as before.



13.1 Magnetic Force on a Charged Particle

$$\underline{\mathbf{F}} = q\,\underline{\mathbf{v}} \times \underline{\mathbf{B}}$$

- $\underline{\mathbf{F}}$ is the magnetic force
- q is the charge
- $\underline{\mathbf{v}}$ is the velocity of the moving charge
- $\underline{\mathbf{B}}$ is the magnetic field



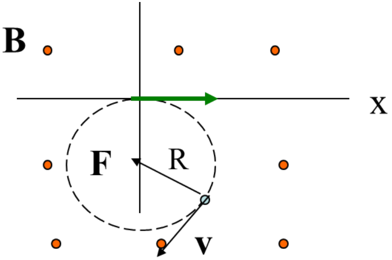
Because the magnetic force is perpendicular to the displacement ($dW = \underline{\mathbf{F}}.\underline{\mathbf{dx}}$), the force does no work on the particle

- Kinetic energy does not change
- Speed does not change
- Only direction changes
- Particle moves in a circle if $v \perp B$

$\underline{\mathbf{F}} = q\,\underline{\mathbf{v}} \times \underline{\mathbf{B}}$

Newton's second law in components

- $m\ddot{x} = F_x$
- $m\ddot{y} = F_y$
- $m\ddot{z} = F_z$



Simple case: $\underline{\mathbf{v}}$ perpendicular to $\underline{\mathbf{B}}$

- Magnetic force= centripetal force
- $F = qvB = \frac{mv^2}{R}$ (magnitudes)

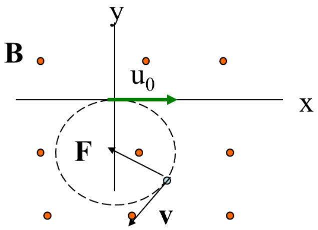
  where R is the radius of curvature
- $R = \frac{mv}{qB} = \frac{p}{qB}$

  p is the particle momentum

13.2 B Field only, $\underline{\mathbf{v}} \perp \underline{\mathbf{B}}$

$$m\underline{\ddot{\mathbf{r}}} = q\underline{\mathbf{v}} \times \underline{\mathbf{B}}$$

- $m\ddot{x} = F_x$, $m\ddot{y} = F_y$, $m\ddot{z} = F_z$
- $\underline{\mathbf{B}}$–field only $\rightarrow$ $\underline{\mathbf{B}} = B_z\underline{\mathbf{k}}$
- $\underline{\mathbf{v}} = (\dot{x}, \dot{y}, 0)$



$$\underline{\mathbf{v}} \times \underline{\mathbf{B}} = \begin{vmatrix} \underline{\mathbf{i}} & \underline{\mathbf{j}} & \underline{\mathbf{k}} \\ \dot{x} & \dot{y} & \dot{z} \\ 0 & 0 & B_z \end{vmatrix} = \dot{y}B_z\underline{\mathbf{i}} - \dot{x}B_z\underline{\mathbf{j}}$$

- $m\ddot{x} = qB_z\dot{y}$; $m\ddot{y} = -qB_z\dot{x}$
- $\ddot{x} = \omega\dot{y}$; $\ddot{y} = -\omega\dot{x}$ where $\omega = \frac{qB_z}{m}$
- $\dot{y} = -\omega x + c$; $(\dot{y} = 0 \text{ at } t = 0 \rightarrow c = 0)$
- $\ddot{x} + \omega^2 x = 0 \rightarrow x = A_1 \cos\omega t + A_2 \sin\omega t$
  - At $t = 0$, $x = 0 \rightarrow A_1 = 0 \rightarrow x = A_2 \sin\omega t$
  - $\dot{x} = A_2\,\omega\cos\omega t$; at $t = 0$, $\dot{x} = u_0 \rightarrow A_2 = \frac{u_0}{\omega}$

- At $t = 0$:
- $\underline{\mathbf{r}}_0 = (0, 0, 0)$
- $\underline{\mathbf{v}}_0 = (u_0, 0, 0)$

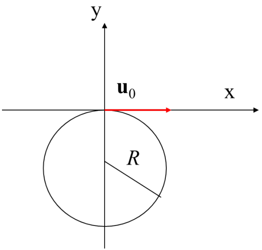
B Field only, continued

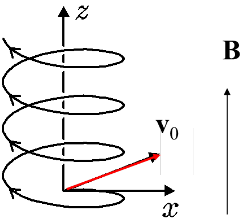
$$x = \frac{u_0}{\omega} \sin \omega t$$

- From before :
  $\dot{y} = -\omega x = -u_0 \sin \omega t$
- $y = \frac{u_0}{\omega} \cos \omega t + c'$
- At $t = 0, \; y = 0 \;\; \rightarrow \;\; c' = -\frac{u_0}{\omega}$

$$y = \frac{u_0}{\omega} (\cos \omega t - 1)$$

- $x = R \sin \omega t \;\; (R = \frac{u_0}{\omega} = \frac{u_0 m}{qB_z})$
- $y + R = R \cos \omega t$
  $\rightarrow$ Circle $\; x^2 + (y + R)^2 = R^2$
- If at $t = 0, \;\; \underline{\mathrm{v}}_0 = (u_0, 0, w_0)$
  The particle will spiral in circles about the z-direction:



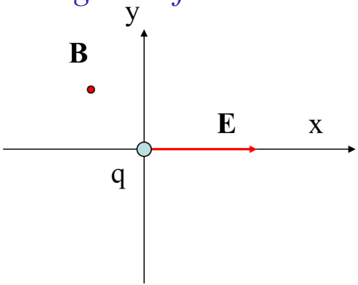


$$z = w_0 t; \;\; x^2 + (y + R)^2 = R^2$$

13.3 Motion under electric and magnetic fields

$$m\underline{\ddot{\mathbf{r}}} = q\underline{\mathbf{v}} \times \underline{\mathbf{B}} + q\underline{\mathbf{E}}$$

- $m\ddot{x} = F_x,\ m\ddot{y} = F_y,\ m\ddot{z} = F_z$
- $\underline{\mathbf{B}}$–field$\rightarrow$ $\underline{\mathbf{B}} = B_z\underline{\mathbf{k}}$
- $\underline{\mathbf{E}}$–field$\rightarrow$ $\underline{\mathbf{E}} = E_x\underline{\mathbf{i}}$



$$\underline{\mathbf{v}} \times \underline{\mathbf{B}} = \begin{vmatrix} \underline{\mathbf{i}} & \underline{\mathbf{j}} & \underline{\mathbf{k}} \\ \dot{x} & \dot{y} & \dot{z} \\ 0 & 0 & B_z \end{vmatrix} = \dot{y}B_z\underline{\mathbf{i}} - \dot{x}B_z\underline{\mathbf{j}}$$

- $m\ddot{x} = qB_z\dot{y} + qE_x;\qquad m\ddot{y} = -qB_z\dot{x}$
- $\ddot{x} = \omega\dot{y} + \frac{qE_x}{m}$ where $\omega = \frac{qB_z}{m}$
- $\dot{y} = -\omega x$ (as before $\dot{y} = 0$ at $t = 0$)
- $\ddot{x} + \omega^2 x = \frac{qE_x}{m}$: solution $x = x_1 + x_2$
  - Complementary function : $x_1 = A_1 \cos\omega t + A_2 \sin\omega t$
  - Particular integral : $x_2 = \frac{qE_x}{m\omega^2}$

- At $t = 0$:
- $\underline{\mathbf{r}}_0 = (0, 0, 0)$
- $\underline{\mathbf{v}}_0 = (0, 0, 0)$

Electric and magnetic fields, continued

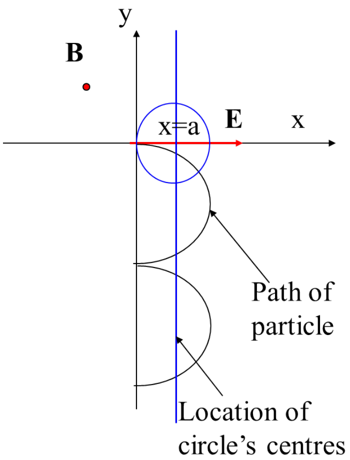
- $x = A_1 \cos \omega t + A_2 \sin \omega t + \frac{qE_x}{m\omega^2}$

  Define $a = \frac{qE_x}{m\omega^2}$
- $t = 0,\ x = 0 \ \rightarrow \ A_1 + a = 0$
- $t = 0,\ \dot{x} = 0 \ \rightarrow \ A_2 = 0$

  $$x = a\,(1 - \cos \omega t)$$
- From before :

  $\dot{y} = -\omega x = -a\omega\,(1 - \cos \omega t)$
- $y = a\,(\sin \omega t - \omega t) + c$
- At $t = 0,\ y = 0 \ \rightarrow \ c = 0$

  $$y = a\,(\sin \omega t - \omega t)$$
- $-a \cos \omega t = x - a$
- $a \sin \omega t = y + a\omega t$



$\rightarrow$ A circle rolling down the $-y$ axis : $(x-a)^2 + (y + a\omega t)^2 = a^2$

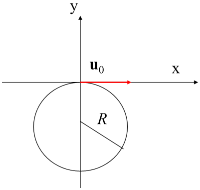
13.4 Kinetic energy in E & B fields

13.4.1 B only, $v \perp B$

- $x = \frac{u_0}{\omega}\sin\omega t \;\rightarrow\; \dot{x} = u_0\cos\omega t$
- $y = \frac{u_0}{\omega}(\cos\omega t - 1)$

  $\rightarrow\; \dot{y} = -u_0\sin\omega t$
- $T = \frac{1}{2}m[u_0\cos\omega t]^2 + \frac{1}{2}m[-u_0\sin\omega t]^2$

  $= \frac{1}{2}mu_0^2 \;\rightarrow$ No energy change

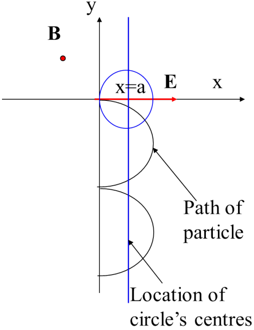


13.4.2 B and E

- $x = a(1 - \cos\omega t) \;\rightarrow\; \dot{x} = a\omega\sin\omega t$
- $y = a(\sin\omega t - \omega t)$

  $\rightarrow\; \dot{y} = a\omega(\cos\omega t - 1)$
- $T = \frac{1}{2}m(a\omega)^2(1 + 1 - 2\cos\omega t)$

  $= \frac{1}{2}m(2a\omega)^2\sin^2\frac{\omega t}{2} \;\rightarrow$ W.D. by $\underline{\mathbf{E}}$ field



13.5 Cyclotron motion (E & B fields)

- Let the electric field vary with time as:

$$\underline{\mathbf{E}} = E_0 \begin{pmatrix} \cos\omega t \\ -\sin\omega t \\ 0 \end{pmatrix}, \quad \underline{\mathbf{B}} = B_z \underline{\mathbf{k}}$$

- Can show by direct substitution

$$x(t) = R\left[\omega\, t \sin\omega t + \sin\omega t - 1\right]$$

$$y(t) = R\left[\omega\, t \cos\omega t - \sin\omega t\right]$$

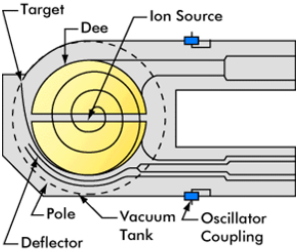
is a solution of the EOM

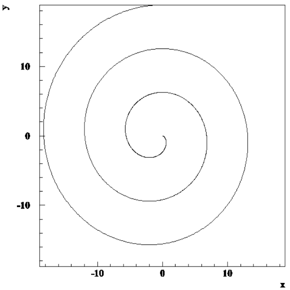
where $R = \frac{qE_0}{m\omega^2}$ and $\omega = \frac{qB_z}{m}$

ω is the Cyclotron frequency

- $(x+R)^2 + y^2 = R^2\left[(\omega t)^2 + 1\right]$
- $T = \frac{1}{2}m[\dot{x}^2 + \dot{y}^2] = \frac{1}{2}mR^2\omega^4 t^2$

$\rightarrow$ particle accelerator





14.1 Differentiation of vectors wrt time

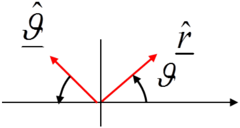
Vectors follow the rules of differentiation:

- $\frac{d}{dt}\underline{\mathbf{a}} = \frac{da_x}{dt}\underline{\mathbf{i}} + \frac{da_y}{dt}\underline{\mathbf{j}} + \frac{da_z}{dt}\underline{\mathbf{k}} = \dot{a}_x\underline{\mathbf{i}} + \dot{a}_y\underline{\mathbf{j}} + \dot{a}_z\underline{\mathbf{k}}$
- $\frac{d}{dt}(\underline{\mathbf{a}} + \underline{\mathbf{b}}) = \frac{d\underline{\mathbf{a}}}{dt} + \frac{d\underline{\mathbf{b}}}{dt} = \dot{\underline{\mathbf{a}}} + \dot{\underline{\mathbf{b}}}$
- $\frac{d}{dt}(c\,\underline{\mathbf{a}}) = \frac{d\,c}{dt}\,\underline{\mathbf{a}} + c\,\frac{d\underline{\mathbf{a}}}{dt} = \dot{c}\,\underline{\mathbf{a}} + c\,\dot{\underline{\mathbf{a}}}$
- $\frac{d}{dt}(\underline{\mathbf{a}}.\underline{\mathbf{b}}) = \frac{d\underline{\mathbf{a}}}{dt}.\underline{\mathbf{b}} + \underline{\mathbf{a}}.\frac{d\underline{\mathbf{b}}}{dt} = \dot{\underline{\mathbf{a}}}.\underline{\mathbf{b}} + \underline{\mathbf{a}}.\dot{\underline{\mathbf{b}}}$
- $\frac{d}{dt}(\underline{\mathbf{a}} \times \underline{\mathbf{b}}) = \frac{d\underline{\mathbf{a}}}{dt} \times \underline{\mathbf{b}} + \underline{\mathbf{a}} \times \frac{d\underline{\mathbf{b}}}{dt} = \dot{\underline{\mathbf{a}}} \times \underline{\mathbf{b}} + \underline{\mathbf{a}} \times \dot{\underline{\mathbf{b}}}$ (order is impt.)

Orthogonality of differentiated unit vectors

- $\frac{d}{dt}(\underline{\hat{\mathbf{r}}}.\underline{\hat{\mathbf{r}}}) = 2\underline{\hat{\mathbf{r}}}.\frac{d\underline{\hat{\mathbf{r}}}}{dt} = 0$ (since $\underline{\hat{\mathbf{r}}}.\underline{\hat{\mathbf{r}}} = 1$)

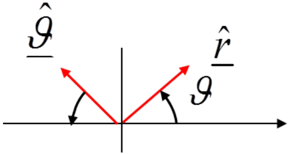
  Therefore $\frac{d\underline{\hat{\mathbf{r}}}}{dt} \perp \underline{\hat{\mathbf{r}}} \rightarrow \frac{d\underline{\hat{\mathbf{r}}}}{dt} \propto \underline{\hat{\theta}}$

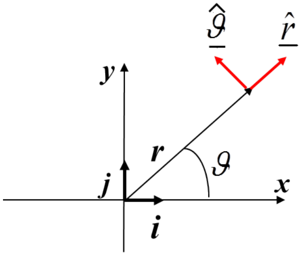


Derivative of any unit vector gives a vector perpendicular to it.

14.1.1 The position vector in polar coordinates

- $\underline{\mathbf{r}} = r_0(\underline{\mathbf{i}} \cos\theta + \underline{\mathbf{j}} \sin\theta)$
  $\hat{\underline{\mathbf{r}}} = (\underline{\mathbf{i}} \cos\theta + \underline{\mathbf{j}} \sin\theta)$ is a unit vector in the direction of $\underline{\mathbf{r}}$
- $\frac{d\hat{\underline{\mathbf{r}}}}{dt} = \left[-\underline{\mathbf{i}} \sin\theta + \underline{\mathbf{j}} \cos\theta\right] \dot{\theta}$
- $\hat{\underline{\theta}} = (-\underline{\mathbf{i}} \sin\theta + \underline{\mathbf{j}} \cos\theta)$ is a unit vector perpendicular to $\underline{\mathbf{r}}$
- $\dot{\hat{\underline{\mathbf{r}}}} = \dot{\theta}\,\hat{\underline{\theta}}$ also $\dot{\hat{\underline{\theta}}} = -\dot{\theta}\,\hat{\underline{\mathbf{r}}}$

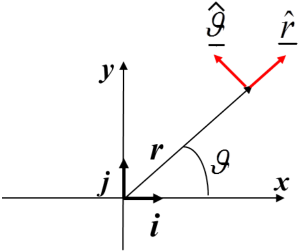




14.1.2 The velocity vector in polar coordinates

- $\underline{\mathbf{r}} = r\,\underline{\hat{\mathbf{r}}}$

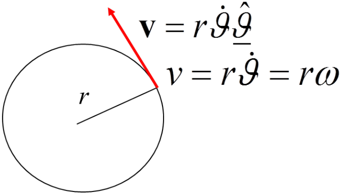
  $\underline{\mathbf{v}} = \underline{\dot{\mathbf{r}}} = \dot{r}\,\underline{\hat{\mathbf{r}}} + r\,\underline{\dot{\hat{\mathbf{r}}}}$
- From before $\underline{\dot{\hat{\mathbf{r}}}} = \dot{\theta}\,\underline{\hat{\theta}}$

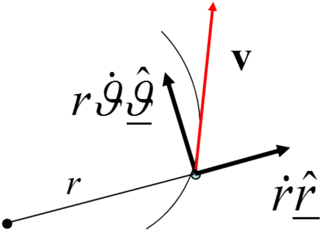


General case: $\underline{\mathbf{v}} = \underline{\dot{\mathbf{r}}} = \dot{r}\,\underline{\hat{\mathbf{r}}} + r\,\dot{\theta}\,\underline{\hat{\theta}}$

For circular motion:

- Since $\dot{r} = 0$
- $\underline{\mathbf{v}} = r\,\dot{\theta}\,\underline{\hat{\theta}} = r\,\omega\,\underline{\hat{\theta}}$





14.1.3 The acceleration vector in polar coordinates

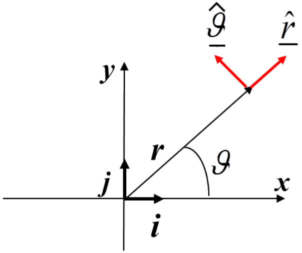
- From before $\underline{\mathbf{v}} = \underline{\dot{\mathbf{r}}} = \dot{r}\,\underline{\hat{\mathbf{r}}} + r\,\dot{\theta}\,\underline{\hat{\theta}}$
- $\underline{\mathbf{a}} = \underline{\dot{\mathbf{v}}} = \underline{\ddot{\mathbf{r}}}$

  $\frac{d}{dt}(\dot{r}\,\underline{\hat{\mathbf{r}}}) = \ddot{r}\underline{\hat{\mathbf{r}}} + \dot{r}\dot{\theta}\underline{\hat{\theta}}$ (since $\underline{\dot{\hat{\mathbf{r}}}} = \dot{\theta}\,\underline{\hat{\theta}}$)

  $\frac{d}{dt}(r\,\dot{\theta}\,\underline{\hat{\theta}}) = r\,\dot{\theta}\,\underline{\dot{\hat{\theta}}} + r\,\ddot{\theta}\,\underline{\hat{\theta}} + \dot{r}\,\dot{\theta}\,\underline{\hat{\theta}}$

  $= -r\,\dot{\theta}^2\,\underline{\hat{\mathbf{r}}} + r\,\ddot{\theta}\,\underline{\hat{\theta}} + \dot{r}\,\dot{\theta}\,\underline{\hat{\theta}}$

  (since $\underline{\dot{\hat{\theta}}} = -\dot{\theta}\,\underline{\hat{\mathbf{r}}}$)



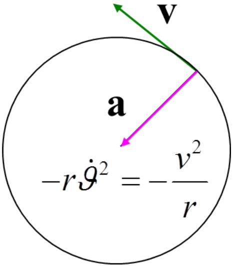
General case :

$$\underline{\mathbf{a}} = \underline{\ddot{\mathbf{r}}} = (\ddot{r} - r\,\dot{\theta}^2)\,\underline{\hat{\mathbf{r}}} + (2\dot{r}\,\dot{\theta} + r\,\ddot{\theta})\,\underline{\hat{\theta}}$$

For circular motion:

- Since scalars $\ddot{r} = \dot{r} = \ddot{\theta} = 0$

  (no change in magnitudes of radius or azimuthal acceleration)

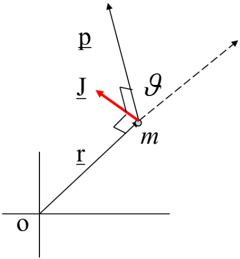


$$\underline{\mathbf{a}} = -r\,\dot{\theta}^2\,\underline{\hat{\mathbf{r}}} = -\omega^2 r\,\underline{\hat{\mathbf{r}}} = -\frac{v^2}{r}\,\underline{\hat{\mathbf{r}}}$$

14.2 Angular momentum and torque

- The definition of *angular momentum* (or the *moment of momentum*) $\underline{\mathbf{J}}$ for a single particle : $\underline{\mathbf{J}} = \underline{\mathbf{r}} \times \underline{\mathbf{p}}$

  $\underline{\mathbf{r}}$ is the displacement vector from the origin and $\underline{\mathbf{p}}$ the momentum
- The direction of the angular momentum gives the direction perpendicular to the plane of motion



  - Differentiate: $\frac{d\underline{\mathbf{J}}}{dt} = \underline{\mathbf{r}} \times \frac{d\underline{\mathbf{p}}}{dt} + \frac{d\underline{\mathbf{r}}}{dt} \times \underline{\mathbf{p}}$
  - Definitions of force and velocity: $\underline{\mathbf{F}} = \frac{d\underline{\mathbf{p}}}{dt}$ and $\underline{\mathbf{v}} = \frac{d\underline{\mathbf{r}}}{dt}$
  - $\frac{d\underline{\mathbf{J}}}{dt} = \underline{\mathbf{r}} \times \underline{\mathbf{F}} + \underline{\mathbf{v}} \times \underline{\mathbf{p}}$ $\leftarrow$ this term$= m\underline{\mathbf{v}} \times \underline{\mathbf{v}} = 0$
  - Define torque $\underline{\tau} = \underline{\mathbf{r}} \times \underline{\mathbf{F}} = \frac{d\underline{\mathbf{J}}}{dt}$ (cf. Linear motion $\underline{\mathbf{F}} = \frac{d\underline{\mathbf{p}}}{dt}$)
  - For multiple forces : $\frac{d\underline{\mathbf{J}}}{dt} = \sum_{i=1}^{n} \underline{\mathbf{r}}_i \times \underline{\mathbf{F}}_i = \underline{\tau}_{tot}$

Torque depends on the origin

- Torque wrt origin O

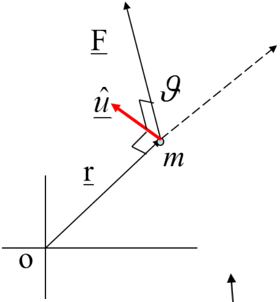
  $\underline{\tau}_o = \underline{\mathbf{r}} \times \underline{\mathbf{F}}$

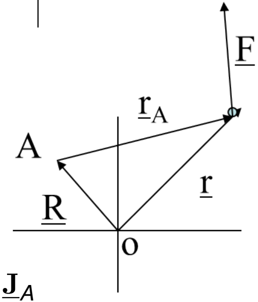
- Torque wrt point A

  $\underline{\tau}_A = \underline{\mathbf{r}}_A \times \underline{\mathbf{F}} = \underline{\mathbf{r}} \times \underline{\mathbf{F}} - \underline{\mathbf{R}} \times \underline{\mathbf{F}}$

  $= \underline{\tau}_0 - \underline{\mathbf{R}} \times \underline{\mathbf{F}}$

- Hence in general $\underline{\tau}_o \neq \underline{\tau}_A$



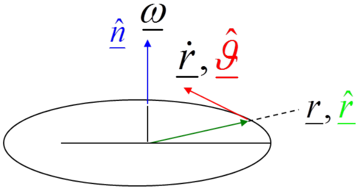


Same applies to angular momentum : $\underline{\mathbf{J}}_o \neq \underline{\mathbf{J}}_A$

14.3 Angular velocity $\underline{\omega}$ for rotation in a circle

- Definition of angular velocity :

$$\underline{\dot{\mathbf{r}}} = \underline{\omega} \times \underline{\mathbf{r}}$$



- Note that $\underline{\dot{\mathbf{r}}}$ is *always* $\perp$ $\underline{\mathbf{r}}$, so $\underline{\omega}$ is defined for circular motion
- Define $\underline{\hat{\mathbf{n}}}$ such that $\underline{\hat{\theta}} = \underline{\hat{\mathbf{n}}} \times \underline{\hat{\mathbf{r}}}$
- Recall $\underline{\mathbf{v}} = \underline{\dot{\mathbf{r}}} = \dot{r}\underline{\mathbf{r}} + r\dot{\theta}\underline{\hat{\theta}}$
- For circular motion $\dot{r} = 0$; $\dot{\theta} = \omega$

$\rightarrow \quad \underline{\dot{\mathbf{r}}} = \underline{\omega} \times \underline{\mathbf{r}} = (\omega\,\underline{\hat{\mathbf{n}}}) \times (r\,\underline{\hat{\mathbf{r}}}) = r\omega\,\underline{\hat{\theta}}$

Relationship between $\underline{\mathbf{J}}$ and $\underline{\omega}$

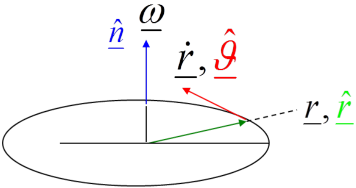
- $\underline{\mathbf{J}} = \underline{\mathbf{r}} \times \underline{\mathbf{p}} = m\,\underline{\mathbf{r}} \times \underline{\dot{\mathbf{r}}} = m\,\underline{\mathbf{r}} \times (\underline{\omega} \times \underline{\mathbf{r}})$
- Recall vector identity $\underline{\mathbf{a}} \times (\underline{\mathbf{b}} \times \underline{\mathbf{c}}) = (\underline{\mathbf{a}}.\underline{\mathbf{c}})\,\underline{\mathbf{b}} - (\underline{\mathbf{a}}.\underline{\mathbf{b}})\,\underline{\mathbf{c}}$
- $\underline{\mathbf{J}} = m\,r^2\,\underline{\omega} - m\,(\underline{\mathbf{r}} \cdot \underline{\omega})\,\underline{\mathbf{r}}$
- $\underline{\mathbf{r}} \cdot \underline{\omega} = 0$ since the circular rotation is in a plane
- Hence $\underline{\mathbf{J}} = \mathtt{I}\underline{\omega}$ where $\mathtt{I} = mr^2$; (generally $\mathtt{I} = \sum_i [m_i r_i^2]$)

15.1 Angular acceleration $\underline{\alpha}$ for rotation in a circle

Angular velocity for rotation in a circle : $\dot{\underline{\mathbf{r}}} = \underline{\omega} \times \underline{\mathbf{r}}$

- $\underline{\omega} = \omega\, \underline{\hat{\mathbf{n}}} = \dot{\theta}\, \underline{\hat{\mathbf{n}}}$
- Angular acceleration:

$$\underline{\alpha} = \dot{\underline{\omega}}$$



Special case if $\underline{\alpha}$ is constant $\rightarrow$

- $\frac{d\omega}{dt} = \alpha \;\rightarrow\; \omega = \omega_0 + \alpha\, t$
- $\frac{d\theta}{dt} = \omega \;\rightarrow\; \theta = \theta_0 + \omega_0\, t + \frac{1}{2}\alpha\, t^2$

  Which should be recognisable equations !

Relationship between $\underline{\tau}$ and $\underline{\alpha}$ for rotation in a circle

$$\underline{\tau} = \frac{d}{dt}\underline{\mathbf{J}} = \mathtt{I}\underline{\alpha}$$

15.2 Angular motion : work and power

- Work linear motion :

  $dW = \underline{\mathbf{F}} \cdot d\underline{\mathbf{s}}$

  $\rightarrow$ $W = \int \underline{\mathbf{F}} \cdot d\underline{\mathbf{s}}$

- Work angular motion :

  $\underline{\tau} = \underline{\mathbf{r}} \times \underline{\mathbf{F}}$

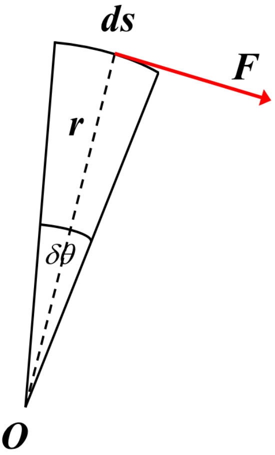
  $d\underline{\mathbf{s}} = d\underline{\theta} \times \underline{\mathbf{r}}$ ($d\underline{\theta}$ out of page)

  $dW = \underline{\mathbf{F}} \cdot d\underline{\mathbf{s}} = \underline{\mathbf{F}} \cdot (d\underline{\theta} \times \underline{\mathbf{r}})$

  $= (\underline{\mathbf{r}} \times \underline{\mathbf{F}}) \cdot d\underline{\theta}$

  (scalar triple product)

  $W = \int \underline{\tau} d\underline{\theta} = \int \underline{\tau} \cdot \underline{\omega}\, dt$



- Power :

  Linear motion : $P = \underline{\mathbf{F}} \cdot \underline{\mathbf{v}}$

  Rotational motion : $P = \underline{\tau} \cdot \underline{\omega}$

15.3 Correspondence between linear and angular quantities

Linear quantities are re-formulated in a rotating frame:

Linear/ translational quantities	Angular/ rotational quantities
Displacement, position: $\underline{\mathbf{r}}$ [m]	Angular displacement, angle: θ [rad]
Velocity: $\underline{\mathbf{v}}$ [m s^{-1}]	Angular velocity: $\underline{\omega}$ [rad s^{-1}]
Acceleration: $\underline{\mathbf{a}}$ [m s^{-2}]	Angular acceleration: $\underline{\alpha}$ [rad s^{-2}]
Mass m [kg]	Moment of inertia: $\mathtt{I}$ [kg m^2 rad^{-1}]
Momentum: $\underline{\mathbf{p}}$ [kg m s]$^{-1}$	Angular momentum: $\underline{\mathbf{J}}$ [kg m^2 s^{-1}]
Force $\underline{\mathbf{F}}$ [N = kg m s^{-2}]	Torque: $\underline{\tau}$ [kg m^2 s^{-2} rad^{-1}]
Weight F_g [N]	Moment [N m]
Work $dW = F \cdot dx$ [N m]	Work $W = \tau \cdot d\theta$ [N m]

15.3.1 Reformulation of Newton's laws for angular motion

1. In the absence of a net applied torque, the angular velocity remains unchanged.

2. Torque = [moment of inertia]$\times$ [angular acceleration]
   $\underline{\tau} = \mathtt{I}\underline{\alpha}$

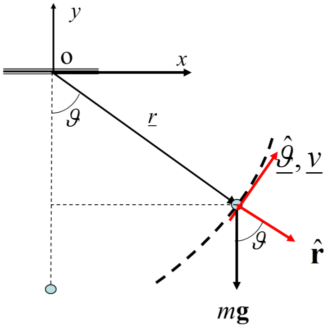
   This expression applies to rotation about a single principal axis, usually the axis of symmetry.
   (cf. $\underline{\mathbf{F}} = m\underline{\mathbf{a}}$). More on moment of inertia comes later.

3. For every applied torque, there is an equal and opposite reaction torque. (A result of Newton's 3rd law of linear motion.)

15.3.2 Example: the simple pendulum

Derive the EOM of a simple pendulum using angular variables:

- $\underline{\tau} = \underline{\mathbf{r}} \times \underline{\mathbf{F}} = -mgr\sin\theta\,\hat{\underline{\mathbf{z}}}$
- $\underline{\mathbf{J}} = \underline{\mathbf{r}} \times m\underline{\mathbf{v}} = mrv\,\hat{\underline{\mathbf{z}}}$
- $v = r\dot{\theta} \quad \rightarrow \quad \dot{v} = r\ddot{\theta}$
- $\frac{d\underline{\mathbf{J}}}{dt} = mr\dot{v}\,\hat{\underline{\mathbf{z}}} = (mr^2\ddot{\theta})\,\hat{\underline{\mathbf{z}}}$
  (since $\hat{\underline{\mathbf{z}}}$ is a constant vector)
- $\frac{d\underline{\mathbf{J}}}{dt} = \underline{\tau} \;\rightarrow\; mr^2\ddot{\theta} = -mgr\sin\theta$
- $\ddot{\theta} + \frac{g}{r}\sin\theta = 0$



15.4 Moments of forces

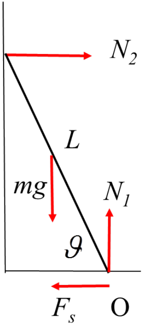
Simple example : ladder against a wall

- If no slipping, torques (moments) must balance
- About any point:

  $\sum_{i=1}^{n} \underline{\mathbf{r}}_i \times \underline{\mathbf{F}}_i = \underline{\tau}_{tot} = 0$
- Moments about O

  $mg\frac{L}{2}\cos\theta = N_2 L \sin\theta$
- Also balance of forces in equilibrium

  $mg = N_1 \; and \; F_s = \mu N_1 = N_2$

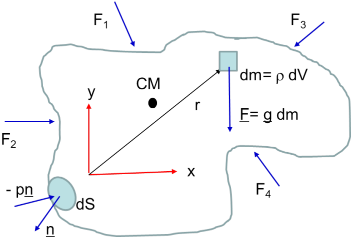


General case: body subject to gravity. Total moment :

- $\underline{\mathbf{M}} = \int_V \underline{\mathbf{r}} \times \underline{\mathbf{g}}\, \rho\, dV$ mass term

  $+\sum_{i=1}^{n} \underline{\mathbf{r}}_i \times \underline{\mathbf{F}}_i$ external forces

  $-\int_S \underline{\mathbf{r}} \times (p\underline{\mathbf{n}}\, dS)$ surface pressure

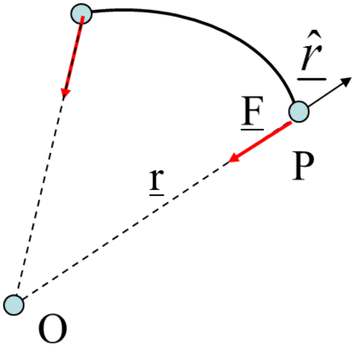


15.5 Central forces

- Central force: $\underline{\mathbf{F}}$ acts towards origin (line joining O and P) always.
- $\underline{\mathbf{F}} = f(r)\,\hat{\underline{\mathbf{r}}}$ only
- Examples:

  Gravitational force $\underline{\mathbf{F}} = -\frac{GmM}{r^2}\hat{\underline{\mathbf{r}}}$

  Electrostatic force $\underline{\mathbf{F}} = \frac{q_1 q_2}{4\pi\epsilon_0 r^2}\hat{\underline{\mathbf{r}}}$



15.5.1 A central force is conservative

A force $\underline{\mathbf{F}}$ is conservative if it meets 3 *equivalent* conditions:

1. The curl of $\underline{\mathbf{F}}$ is zero : $\nabla \times \underline{\mathbf{F}} = 0$
2. Work over closed path $W \equiv \oint_C \underline{\mathbf{F}} \cdot \mathrm{d}\underline{\mathbf{r}} = 0$, independent of path
3. $\underline{\mathbf{F}}$ can be written in terms of scalar potential $\underline{\mathbf{F}} = -\nabla U$

- Equivalence of 1 & 2 from Stokes' theorem $\int_S (\nabla \times \underline{\mathbf{F}}) \cdot \mathrm{d}\underline{\mathbf{a}} = \oint_C \underline{\mathbf{F}} \cdot \mathrm{d}\underline{\mathbf{r}} = 0$
- Equivalence of 1 & 3 from vector calculus identity : $\nabla \times (\nabla U) = 0$

For a *central* potential, take the grad of $U(r)$:

- In cartesians $\nabla U(r) = \frac{\partial U(\sqrt{x^2+y^2+z^2})}{\partial x}\,\hat{\underline{\mathbf{x}}} + \ldots$ ($\hat{\underline{\mathbf{y}}}$ and $\hat{\underline{\mathbf{z}}}$ terms)
- Chain rule $\frac{\partial U}{\partial x} = \frac{\partial U}{\partial r}\frac{\partial r}{\partial x}$: $\nabla U(r) = \frac{x}{\sqrt{x^2+y^2+z^2}}\frac{\partial U(r)}{\partial r}\,\hat{\underline{\mathbf{x}}} + \ldots$
- Since $\frac{x\hat{\underline{\mathbf{x}}}+y\hat{\underline{\mathbf{y}}}+z\hat{\underline{\mathbf{z}}}}{\sqrt{x^2+y^2+z^2}} = \hat{\underline{\mathbf{r}}} \rightarrow -\nabla U(r) = -\frac{\partial U(r)}{\partial r}\hat{\underline{\mathbf{r}}} \equiv f(r)\hat{\underline{\mathbf{r}}} = \underline{\mathbf{F}}(\underline{\mathbf{r}})$

The grad of the scalar potential has only one non-vanishing component which is along $\hat{\underline{\mathbf{r}}}$ ($\rightarrow$ central force).
Hence condition (3) is satisfied $\rightarrow$ central force is conservative force.

16.1 Central force : the equation of motion

- Recall the acceleration in polar coordinates

$$\underline{\mathbf{a}} = \underline{\ddot{\mathbf{r}}} = (\ddot{r} - r\,\dot{\theta}^2)\,\underline{\hat{\mathbf{r}}} + (2\dot{r}\,\dot{\theta} + r\,\ddot{\theta})\,\underline{\hat{\theta}}$$

- If $\underline{\mathbf{F}} = f(r)\,\underline{\hat{\mathbf{r}}}$ only, then $F_\theta = 0$

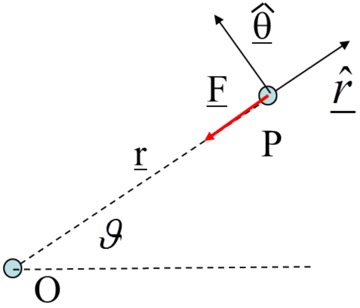
$\rightarrow\ F_\theta = m(2\dot{r}\,\dot{\theta} + r\,\ddot{\theta}) = 0$

$\rightarrow\ F_r = m(\ddot{r} - r\,\dot{\theta}^2) = f(r)$

- Consider $\frac{d}{dt}(r^2\dot{\theta}) = 2r\dot{r}\,\dot{\theta} + r^2\,\ddot{\theta}$

Hence $\frac{1}{r}\,\frac{d}{dt}(r^2\dot{\theta}) = 0$

$\rightarrow\ (r^2\dot{\theta}) =$ constant of motion



- The angular momentum in the plane :

$$\underline{\mathbf{J}} = m\,\underline{\mathbf{r}} \times \underline{\mathbf{v}} = m\,\underline{\mathbf{r}} \times (\dot{r}\,\underline{\hat{\mathbf{r}}} + r\,\dot{\theta}\,\underline{\hat{\theta}}) = (mr^2\,\dot{\theta})\,\underline{\hat{\mathbf{n}}}$$

where $\underline{\mathbf{r}} \times \underline{\hat{\mathbf{r}}} = 0$ and $\underline{\hat{\mathbf{n}}} = \underline{\hat{\mathbf{r}}} \times \underline{\hat{\theta}}$

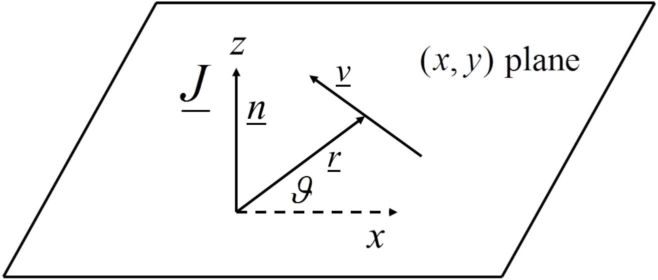
- Torque about origin : $\underline{\tau} = \frac{d\mathbf{J}}{dt} = \underline{\mathbf{r}} \times \underline{\mathbf{F}} = 0$ ($\underline{\mathbf{F}}$ acts along $\underline{\mathbf{r}}$)

Angular momentum vector is a constant of the motion

16.2 Motion under a central force

16.2.1 Motion in a plane

- $\underline{\mathbf{J}} = m\,\underline{\mathbf{r}} \times \underline{\mathbf{v}}$
- Angular momentum is *always* perpendicular to $\underline{\mathbf{r}}$ and $\underline{\mathbf{v}}$
- $\underline{\mathbf{J}}$ is a constant vector ; $\underline{\mathbf{J}} \cdot \underline{\mathbf{r}} = 0$; $\underline{\mathbf{J}} \cdot \underline{\mathbf{v}} = 0$

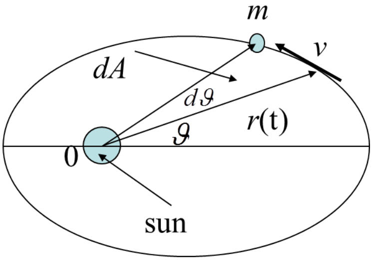


Motion under a central force lies in a plane

16.2.2 Sweeping out equal area in equal time

- Central force example : planetary motion : $|\underline{\mathbf{F}}_r| = \frac{GMm}{r^2}$
- Angular momentum is conserved

  $\rightarrow \; |\underline{\mathbf{J}}| = mr^2\,\dot{\theta} = \text{constant}$



- $dA \approx \frac{1}{2}\,r^2\,d\theta$
- $\frac{dA}{dt} = \frac{1}{2}\,r^2\,\dot{\theta}$

  $\frac{dA}{dt} = \frac{J}{2m} = \text{constant}$ (Kepler 2^{nd} Law)

Orbit sweeps out equal area in equal time

16.3 Central force : the total energy

- Total energy = kinetic + potential :
  $E = T + U(r) = \frac{1}{2}mv^2 + U(r)$ = constant
- $\underline{\mathbf{v}} = \dot{r}\hat{\underline{\mathbf{r}}} + r\dot{\theta}\hat{\underline{\theta}} \rightarrow \ |\underline{\mathbf{v}}|^2 = (\hat{\underline{\mathbf{r}}} + r\dot{\theta}\hat{\underline{\theta}}) \cdot (\hat{\underline{\mathbf{r}}} + r\dot{\theta}\hat{\underline{\theta}})$
  $\rightarrow \ |\underline{\mathbf{v}}|^2 = \dot{r}^2 + r^2\dot{\theta}^2$ (since $\hat{\underline{\mathbf{r}}} \cdot \hat{\underline{\theta}} = 0$)
- $E = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + U(r)$
- No external torque: angular momentum is conserved
  $\rightarrow \ |\underline{\mathbf{J}}| = mr^2\,\dot{\theta} = \text{constant}$

$$E = \frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} + U(r)$$

- Potential energy for a central force
  $U(r) = -\int_{r_{ref}}^{r} \underline{\mathbf{F}} \cdot d\underline{\mathbf{r}} = -\int_{r_{ref}}^{r} f(r)\, dr$

16.3.1 The potential term (inverse square interaction)

- $\underline{\mathbf{F}} = -\frac{A}{r^2}\,\underline{\hat{\mathbf{r}}} \;\rightarrow\; f(r) = -\frac{A}{r^2}$

  [Attractive force for $A > 0 \;\rightarrow$

  signs are important !]

- $U(r) = -\int_{r_{ref}}^{r} \underline{\mathbf{F}} \cdot d\underline{\mathbf{r}}$

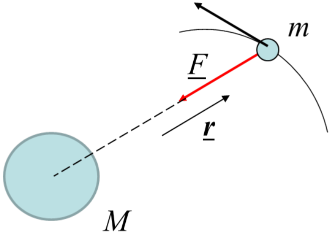
  $= -\int_{r_{ref}}^{r} f(r)\,dr$

- $U(r) = -\frac{A}{r} + \frac{A}{r_{ref}}$

  Usual to define $U(r) = 0$ at

  $r_{ref} = \infty$

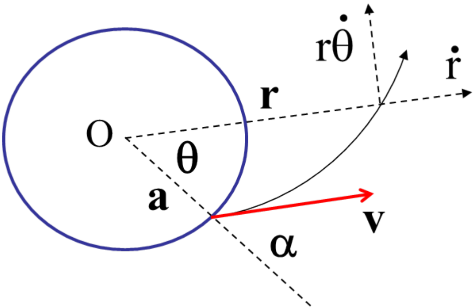
  $\rightarrow\; U(r) = -\frac{A}{r}$



Newton law of gravitation : $\underline{\mathbf{F}} = -\frac{GMm}{r^2}\,\underline{\hat{\mathbf{r}}} \;\rightarrow\; U(r) = -\frac{GMm}{r}$

16.3.2 Example

A projectile is fired from the earth's surface with speed *v* at an angle α to the radius vector at the point of launch. Calculate the projectile's subsequent maximum distance from the earth's surface. Assume that the earth is stationary and its radius is *a*.



16.3.2 Example : solution

- $U(r) = -\frac{GMm}{r}$
- $|\underline{\mathbf{J}}| = m|\underline{\mathbf{r}} \times \underline{\mathbf{v}}| = mav \sin\alpha$
- Energy equation : $E = \frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} + U(r)$

  $\rightarrow \; E = \frac{1}{2}m\dot{r}^2 + \frac{ma^2v^2\sin^2\alpha}{2r^2} - \frac{GMm}{r}$
- At $r = a$: $E = \frac{1}{2}mv^2 - \frac{GMm}{a}$. At maximum height : $\dot{r} = 0$

  $\rightarrow \; \frac{1}{2}mv^2 - \frac{GMm}{a} = \frac{ma^2v^2\sin^2\alpha}{2r_{max}^2} - \frac{GMm}{r_{max}} \qquad (1)$

  $\rightarrow \; \left(v^2 - \frac{2GM}{a}\right) r_{max}^2 + 2GM\, r_{max} - a^2v^2\sin^2\alpha = 0$
- Solve and take the positive root
- Note from Equ.(1) : When $\dot{r} \rightarrow 0$ as $r_{max} \rightarrow \infty$, the rocket *just* escapes the earth's gravitational field

  i.e. $\frac{1}{2}mv^2 - \frac{GMm}{a} \rightarrow 0$, $v_{esc} = \sqrt{\frac{2GM}{a}}$ (independent of α)

17.1 Effective potential

- Energy equation : $E = \frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} + U(r)$
- Define *effective potential* : $U_{eff}(r) = \frac{J^2}{2mr^2} + U(r)$

  $\rightarrow$ then $E = \frac{1}{2}m\dot{r}^2 + U_{eff}(r)$
- Note this has the same form as a 1-D energy expression :

  $\rightarrow$ $E = \frac{1}{2}m\dot{x}^2 + U(x)$

  $\rightarrow$ the analysis becomes 1-D-like problem since $J = \text{const}$
- Allows to predict important features of motion without solving the radial equation

  $\rightarrow$ $\frac{1}{2}m\dot{r}^2 = E - U_{eff}(r)$ $\leftarrow$ LHS is always positive

  $\rightarrow$ $U_{eff}(r) < E$

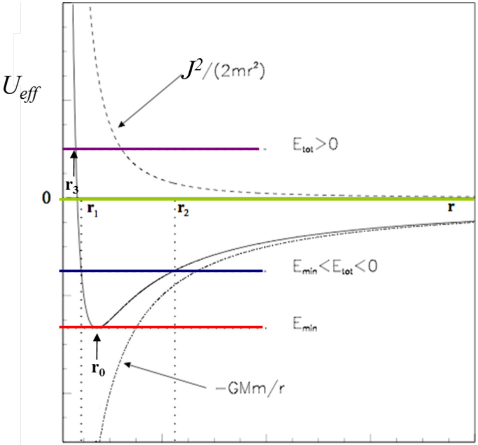
  The only locations where the particle is allowed to go are those with $U_{eff}(r) < E$

17.1.1 $U_{eff}(r)$ for inverse square law

- $U_{eff}(r) = \frac{J^2}{2mr^2} - \frac{GmM}{r}$
- $U_{eff}(r) < E_{tot}$ for all r

Three cases :

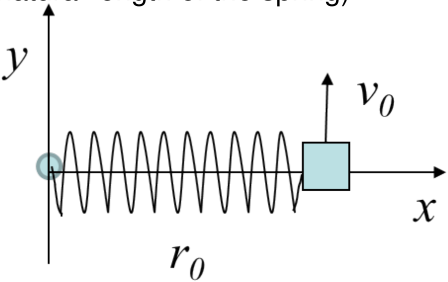
- $E_{tot} < 0$: Bound (closed) orbit with $r_1 < r < r_2$
- E_{tot} has minimum energy at $r = r_0$: $\frac{dU_{eff}}{dr} = 0$, circular motion with $\dot{r} = 0$
- $E_{tot} > 0$: Unbound (open) orbit with $r > r_3$



17.2 Examples

17.2.1 Example 1 : 2-D harmonic oscillator

- $\underline{\mathbf{F}} = -k\underline{\mathbf{r}}$ (ignore the natural length of the spring)



- Energy equation :
  $E = \frac{1}{2}m\dot{r}^2 + U_{eff}(r)$
- $U_{eff}(r) = \frac{J^2}{2mr^2} + \frac{1}{2}kr^2$
- For circular motion $\dot{r} = 0$:
  E_{min} when $\frac{\partial U_{eff}}{\partial r}|_{r_0} = 0$

  $\rightarrow -\frac{J^2}{mr_0^3} + kr_0 = 0$

  where $J = mv_0r_0$
- Leads to $\frac{mv_0^2}{r_0} = k\,r_0$

  as expected

Including the natural length:

- $\underline{\mathbf{F}} = -k\underline{\mathbf{r}} \rightarrow \underline{\mathbf{F}} = -k(\underline{\mathbf{r}} - \underline{\mathbf{a}})$
- $U = \frac{1}{2}kr^2 \rightarrow U = \frac{1}{2}k(r-a)^2$

Leads to $\frac{mv_0^2}{r_0} = k\,(r_0 - a)$

Example continued

- $U_{eff}(r) = \frac{J^2}{2mr^2} + \frac{1}{2}kr^2$
- For general motion :
- $\underline{\mathbf{F}} = -k\underline{\mathbf{r}}$

  $\rightarrow \quad m\ddot{x} = -kx$

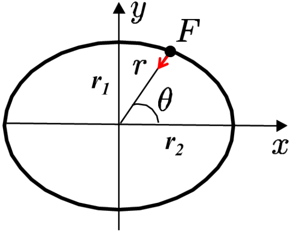
  $\rightarrow \quad m\ddot{y} = -ky$
- Solution for B.C's at $t = 0$:

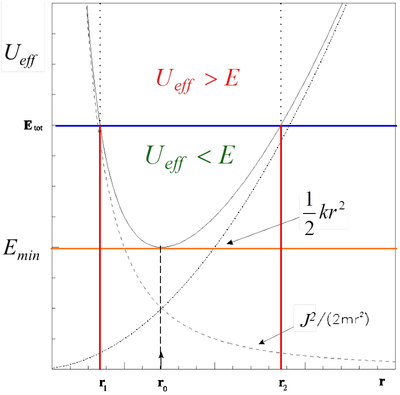
  $x = r_2, y = 0, \dot{x} = 0$

  $\rightarrow \quad x = r_2 \cos \omega t$

  $\rightarrow \quad y = r_1 \sin \omega t$

  where $\omega^2 = \frac{k}{m}$
- Ellipse: $\quad (\frac{x}{r_2})^2 + (\frac{y}{r_1})^2 = 1$

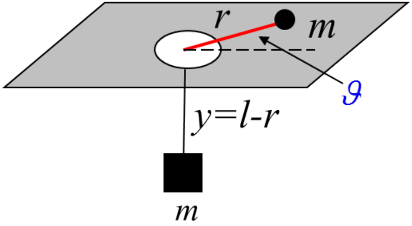




17.2.2 Example 2 : Rotating ball on table

Two particles of mass m are connected by a light inextensible string of length ℓ. The particle on the table starts at $t = 0$ at a distance $\ell/2$ from the hole at a speed v_0 perpendicular to the string. Find the speed at which the particle below the table falls.

- Energy equation :
  $E = \frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} + U(r)$
- $\dot{y} = \dot{r}$, $U(r) = -mgy$
- $E = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}m\dot{y}^2 + $
  $+\frac{J^2}{2mr^2} - mg(\ell - r)$



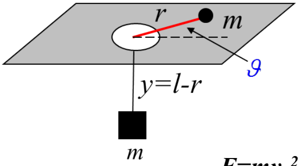
  - At $t = 0$: $J = \frac{mv_0\ell}{2}$, $E = \frac{1}{2}mv_0^2 - mg\frac{\ell}{2}$
  - Solution : $\dot{r}^2 = \frac{g\ell + v_0^2}{2} - \frac{(\ell v_0)^2}{8}\frac{1}{r^2} - gr.$

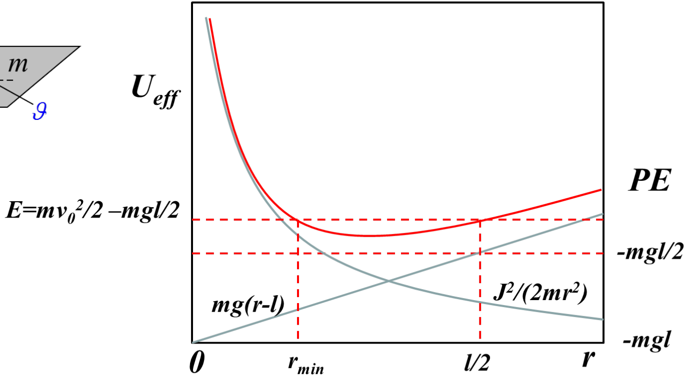
Condition for the particle on the table to move in circular motion

$\rightarrow$ $\dot{r} = 0$, Equate forces $\frac{mv_0^2}{\ell/2} = mg$ $\rightarrow$ gives $\frac{v_0^2}{g\ell} = \frac{1}{2}$

Example 2 continued : effective potential

- Effective potential : $U_{eff} = \frac{J^2}{2mr^2} - mg(\ell - r)$
- Closed orbit with $r_{min} < r < \ell/2$
- Ball never passes though hole in absence of friction, minimum radius $r = r_{min}$





18.1 The orbit equation

Note that the derivation of this is off syllabus

- Acceleration in polar coordinates

  $\underline{\mathbf{a}} = \underline{\ddot{\mathbf{r}}} = (\ddot{r} - r\,\dot{\theta}^2)\,\underline{\hat{\mathbf{r}}} + \frac{1}{r}\,\frac{d}{dt}(r^2\dot{\theta})\,\underline{\hat{\theta}}$

- If $\underline{\mathbf{F}} = f(r)\,\underline{\hat{\mathbf{r}}}$ only, then $F_\theta = 0$. $J = mr^2\dot{\theta} = \text{constant}$.

  $\rightarrow\; F_r = m(\ddot{r} - r\,\dot{\theta}^2) = -\frac{\alpha}{r^2}$ (gravitational force, $\alpha = GmM$)

- Hence $\ddot{r} = \frac{J^2}{m^2\,r^3} - \frac{\alpha}{mr^2} = \frac{u^3\,J^2}{m^2} - \frac{u^2\,\alpha}{m}$ (where $u = \frac{1}{r}$) (1)
- $\dot{r} = \frac{d\theta}{dt}\frac{dr}{d\theta} = \frac{J}{m\,r^2}\frac{dr}{d\theta} = -\frac{J}{m}\frac{d(1/r)}{d\theta} = -\frac{J}{m}\frac{du}{d\theta}$
- $\ddot{r} = \frac{d}{dt}(\dot{r}) = \frac{d\theta}{dt}\frac{d}{d\theta}\left(-\frac{J}{m}\frac{du}{d\theta}\right) = -(\frac{J^2}{m^2}u^2)\frac{d^2u}{d\theta^2}$
- Substituting in Eq (1) : $-(\frac{J^2}{m^2})u^2\frac{d^2u}{d\theta^2} = \frac{u^3\,J^2}{m^2} - \frac{u^2\,\alpha}{m}$

  $\rightarrow\; \frac{d^2u}{d\theta^2} = -u + \frac{m\,\alpha}{J^2} \;\rightarrow\; \boxed{\frac{d^2u}{d\theta^2} = -u + \frac{1}{r_0}}$ $\quad (r_0 = \frac{J^2}{m\,\alpha})$

The orbit equation continued

$$\frac{d^2u}{d\theta^2} = -u + \frac{1}{r_0}$$ where $u = \frac{1}{r}$ and $$r_0 = \frac{J^2}{m\,\alpha}$$

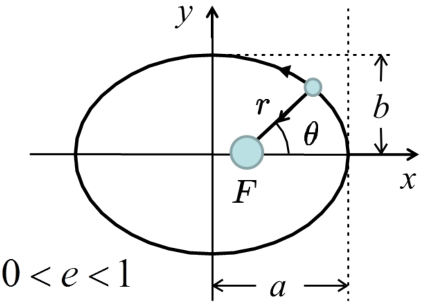
- Solution is $\frac{1}{r} = \frac{1}{r_0} + C\cos(\theta - \theta_0)$ where C, θ_0 = constants

$$r(\theta) = \frac{r_0}{1+e\cos(\theta-\theta_0)}$$ e = eccentricity $(e = C\,r_0)$

- This is in the form of an ellipse†. Also have a link between angular momentum and the ellipse geometry ($J^2 = m\,\alpha r_0$).

† More precisely a conic section, which includes hyperbola, parabola and circle.

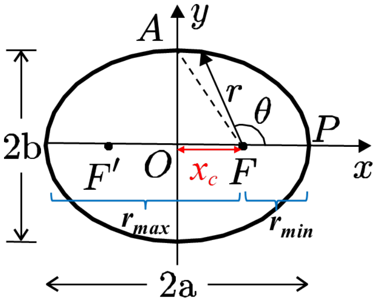
- Choose major axis as x axis $\rightarrow \theta_0 = 0$
- $r(\theta) = \frac{r_0}{1+e\cos\theta}$
- Equivalent form of ellipse : $(\frac{x}{a})^2 + (\frac{y}{b})^2 = 1$
- $b = a\sqrt{(1-e^2)}$
- $a = \frac{r_0}{(1-e^2)}$



Example of a rotating planet : the sun is at the ellipse focus F

$$r(\theta) = \frac{r_0}{1+e\cos\theta} \qquad \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

- Closest approach $\theta = 0$
  "perigee" $r_{min} = \frac{r_0}{1+e}$
  Furthest approach $\theta = \pi$
  "apogee" $r_{max} = \frac{r_0}{1-e}$
  ($\dot{r} = 0$ in both cases)



- $r_{max} + r_{min} = 2a = \frac{2r_0}{1-e^2}$
  $\rightarrow$ $r_0 = a(1-e^2)$
  - $x_c + r_{min} = a \rightarrow x_c + \frac{r_0}{1+e} = a$
    $\rightarrow \quad x_c = a - a(1-e) = a\,e$
  - At point A, $r^2 = x_c^2 + b^2$; $\cos\theta = -\frac{x_c}{r}$; $r = \frac{r_0}{1-ex_c/r}$
    $\rightarrow (r_0 + ex_c)^2 = x_c^2 + b^2 \rightarrow \left(a(1-e^2) + e^2 a\right)^2 = e^2a^2 + b^2$

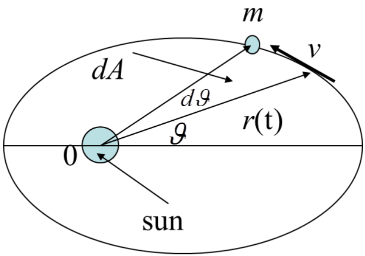
$b = a\sqrt{(1-e^2)}$ also $r_{min} = a(1-e)$ & $r_{max} = a(1+e)$

18.2 Kepler's Laws

- KI: "The orbit of every planet is an ellipse with the sun at one of the foci". [Already derived]
- KII: "A line joining a planet and the sun sweeps out equal areas during equal intervals of time". [Already derived]
- KIII: "The squares of the orbital periods of planets are directly proportional to the cubes of the semi-major axis of the orbits".

18.2.1 Kepler III

"The squares of the orbital periods of planets are directly proportional to the cubes of the semi-major axis of the orbits"



This is trivial for a circle :

$$\rightarrow \quad mr_0\omega^2 = mr_0(\tfrac{2\pi}{T})^2 = \frac{GmM}{r_0^2}$$

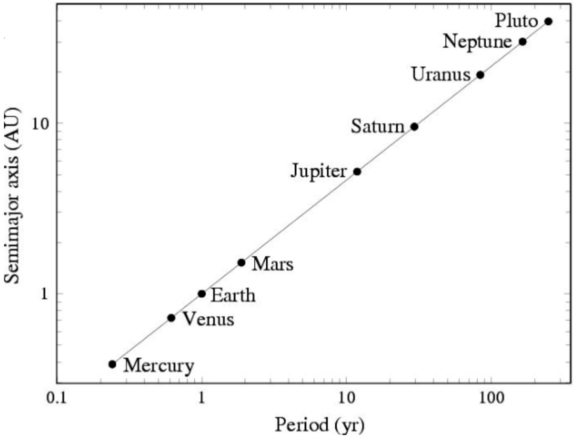
$$\rightarrow r_0^3 = T^2 \frac{GM}{4\pi^2}$$

For an ellipse:

- $\frac{dA}{dt} = \frac{1}{2} r^2 \, \dot{\theta} = \frac{J}{2m} =$ constant
- Integrate $A = \int_0^T \frac{J}{2m} dt \;\rightarrow\; A^2 = (\frac{J}{2m})^2 T^2$
- From before : $r_0 = a(1-e^2)$, $b = a\sqrt{(1-e^2)} \rightarrow a = \frac{b^2}{r_0}$
- Area of an ellipse : $A = \pi ab \;\rightarrow A^2 = \pi^2 a^3 r_0$
- Putting it all together $\rightarrow\;$ $T^2 \propto a^3$

18.2.2 Planetary data

Kepler-III "The squares of the orbital periods of planets are directly proportional to the cubes of the semi-major axis of the orbits"



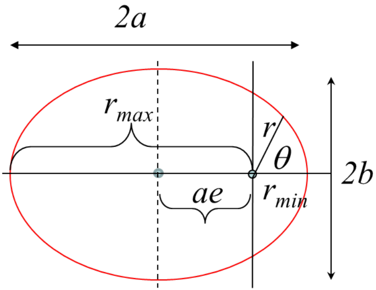
- IMPRESSIVE !

18.3 Elliptical orbit via energy ($E_{min} < E < 0$)

- $E = \frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} - \frac{\alpha}{r}$
- At turning points
  $\dot{r} = 0 \;\rightarrow r = r_{min}$ or $r = r_{max}$
- $E = \frac{J^2}{2mr^2} - \frac{\alpha}{r}$

  $\rightarrow r^2 + \frac{\alpha}{E} r - \frac{J^2}{2mE} = 0$

  $\rightarrow r = -\frac{\alpha}{2E} \pm \left[\left(\frac{\alpha}{2E}\right)^2 + \frac{J^2}{2mE}\right]^{\frac{1}{2}}$
- $r_{min,max} = -\left(\frac{\alpha}{2E}\right)\left[1 \pm \underbrace{\left(1 + \frac{2EJ^2}{m\alpha^2}\right)^{\frac{1}{2}}}_{e}\right]$



$$r_{max} = \underbrace{-\frac{\alpha}{2E}(1 + e)}_{= \text{a}(1 + \text{e})} \;, \quad r_{min} = \underbrace{-\frac{\alpha}{2E}(1 - e)}_{= \text{a}(1 - \text{e})}$$

Consistent with the orbit equations. NICE!

Also:
$(1 + \frac{2EJ^2}{m\alpha^2})^{\frac{1}{2}}) = e$
$E = \frac{m\alpha^2}{2J^2}(e^2 - 1)$
$\quad = \frac{\alpha}{2r_0}(e^2 - 1)$
where $r_0 = \frac{J^2}{m\alpha}$ and
e = eccentricity of ellipse.

Elliptical orbit via energy, continued

$$r(\theta) = \frac{r_0}{1+e\cos\theta}$$

Total energy in ellipse parameters

$$E = \frac{\alpha}{2r_0}\left(e^2 - 1\right)$$

- $e = 0, r = r_0, E = -\frac{\alpha}{2r_0}$

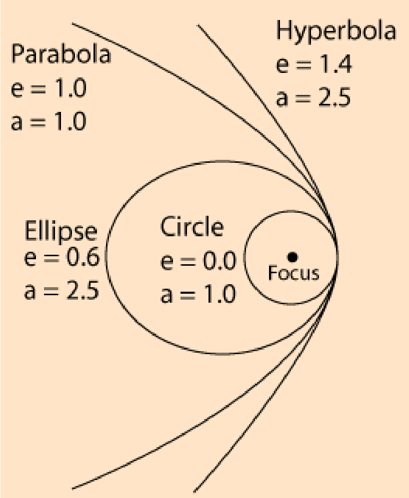
  $\rightarrow$ motion in a circle
- If $0 < e < 1$, $E < 0$

  $\rightarrow$ motion is an ellipse
- If $e = 1$, $E = 0$

  $r(\theta) = \frac{r_0}{1+\cos\theta}$

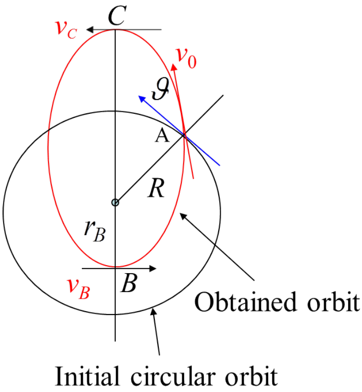
  $\rightarrow$ motion is a parabola
- If $e > 1$, $E > 0$

  $\rightarrow$ motion is a hyperbola



19.1 Example: mistake in the direction of a satellite

- Mistake is made in boosting a satellite, at radius R, into circular orbit : magnitude of velocity is right but direction is wrong.
- Intended to apply thrust to give velocity v_0 along circular orbit.
- Instead thrust at angle θ wrt direction of motion.
- Energy of orbit is right, angular momentum is wrong.



What is the perigee and apogee of the resulting orbit? (Points B & C)

- Conservation of angular momentum, points A & B

$$J = mv_0 R \sin(\tfrac{\pi}{2} - \theta) = mv_B r_B$$

- Energy at A = energy at perigee B

$$\tfrac{1}{2} m v_0^2 - \tfrac{\alpha}{R} = \tfrac{1}{2} m \dot{r}^2 + \frac{J^2}{2 m r_B^2} - \frac{\alpha}{r_B}$$

where $\alpha = GMm$

19.1 Example continued

- At point B, $\dot{r} = 0$, energy conservation becomes

$$\frac{1}{2}mv_0^2 - \frac{\alpha}{R} = \frac{m^2 v_0^2 R^2 \cos^2\theta}{2mr_B^2} - \frac{\alpha}{r_B}$$

- Equate forces for circular motion to get v_0 :

$$\frac{mv_0^2}{R} = \frac{\alpha}{R^2} \quad \rightarrow v_0^2 = \frac{\alpha}{mR}$$

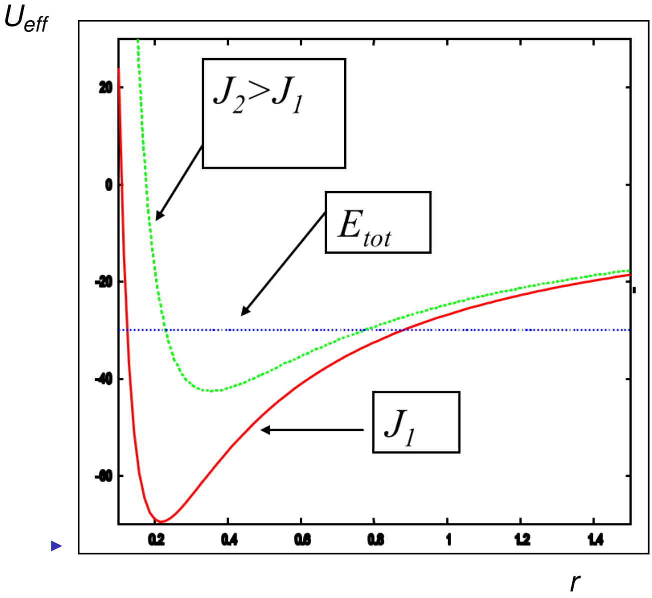
- Sub for v_0^2 : energy conservation becomes

$$\frac{\alpha}{2R} - \frac{\alpha}{R} = \frac{\alpha R\cos^2\theta}{2r_B^2} - \frac{\alpha}{r_B}$$

- $-\frac{1}{2R} = \frac{R\cos^2\theta}{2r_B^2} - \frac{1}{r_B}$
- $(\times 2Rr_B^2) \quad \rightarrow \quad r_B^2 - 2Rr_B + R^2\cos^2\theta = 0$
- $r_B = R - \sqrt{R^2 - R^2\cos^2\theta}$, also $r_C = R + \sqrt{R^2 - R^2\cos^2\theta}$
- $r_B = R(1 - \sin\theta)$ Perigee

  $r_C = R(1 + \sin\theta)$ Apogee

19.1.1 Orbits with the same energy



19.2 Impulse leaving angular momentum unchanged

- Example: A satellite in circular orbit has been given an impulse leaving J unchanged. The kinetic energy is changed by $T = \beta T_0$. Describe the subsequent motion.
- If J is not changed, impulse must be perpendicular to the direction of motion, with angular part of the velocity unchanged.

- $E = \frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} - \frac{\alpha}{r}$
- Circular orbit:

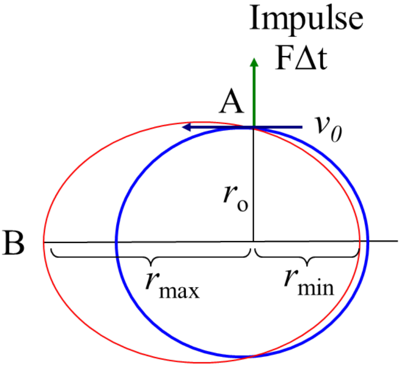
$\rightarrow \dot{r} = 0\ ,\ J = mr_0 v_0 \quad (1)$

$\rightarrow E_{initial} = \frac{1}{2}mv_0^2 - \frac{\alpha}{r_0}$

- Equate forces :

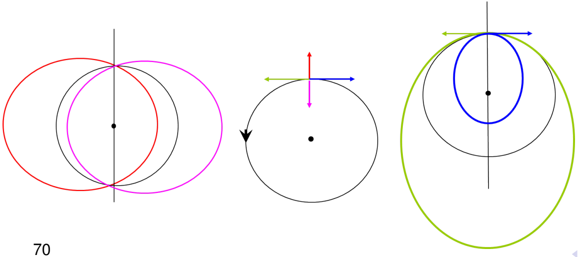
$\rightarrow \frac{mv_0^2}{r_0} = \frac{\alpha}{r_0^2}$

$\rightarrow v_0^2 = \frac{\alpha}{mr_0} \qquad (2)$



19.2 Example continued

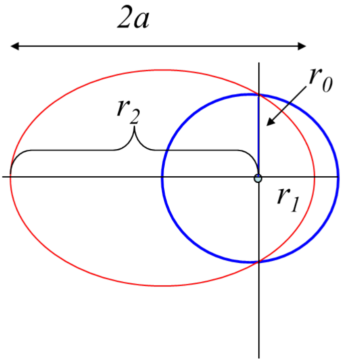
- New orbit (elliptical): $E_{new} = \frac{1}{2}\beta m v_0^2 - \frac{\alpha}{r_0}$
- Equate energies: subsequent motion described by:
- $\frac{1}{2}\beta m v_0^2 - \frac{\alpha}{r_0} = \frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} - \frac{\alpha}{r}$ (3)
- Now solve for r_{min} , r_{max} . Set $\dot{r} = 0$
- From (1), (2), (3) $\rightarrow$ $(\beta - 2)\, r^2 + 2r_0\, r - r_0^2 = 0$
- $r_{min,max} = \frac{-r_0 \pm \sqrt{r_0^2 + (\beta-2)\, r_0^2}}{(\beta-2)}$
- Example: $\beta = 1.001 \rightarrow r_{max} = 1.033\, r_0$ $r_{min} = 0.968\, r_0$

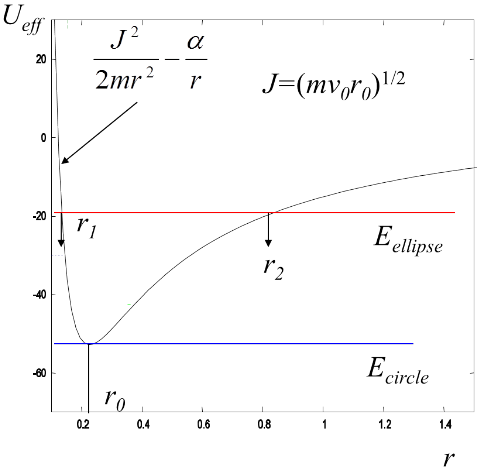


Changes in orbit as a result of impulse

19.2.1 Orbits with the same angular momentum

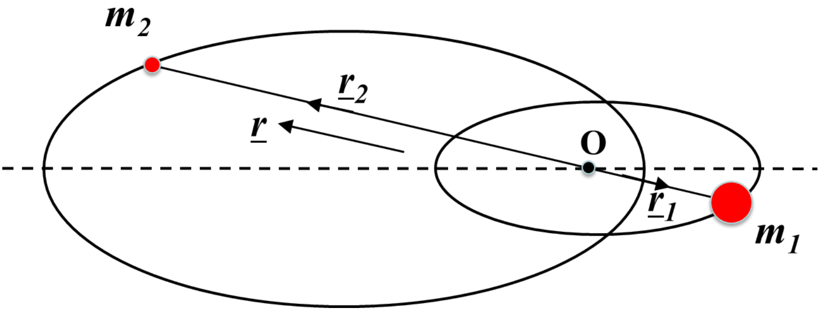
- $E_{ellipse} = -\frac{\alpha}{2\,a}$
- $E_{circle} = -\frac{\alpha}{2\,r_0}$





19.3 Mutual orbits

- The two bodies make a mutual elliptical orbit on either side of the C of M (origin) in a straight line through the C of M
- Relative position vector : $\underline{\mathbf{r}} = \underline{\mathbf{r}_2} - \underline{\mathbf{r}_1}$
- Definition of C of M about O : $m_1 \underline{\mathbf{r}_1} + m_2 \underline{\mathbf{r}_2} = 0$



Mutual orbits continued

- Internal forces: $\underline{\mathbf{F_{12}}} = m_1 \ddot{\underline{\mathbf{r_1}}}$, $\underline{\mathbf{F_{21}}} = m_2 \ddot{\underline{\mathbf{r_2}}}$

  Then $\ddot{\underline{\mathbf{r}}} = \ddot{\underline{\mathbf{r_2}}} - \ddot{\underline{\mathbf{r_1}}} = \frac{\underline{\mathbf{F_{21}}}}{m_2} - \frac{\underline{\mathbf{F_{12}}}}{m_1}$

  But $\underline{\mathbf{F_{12}}} = -\underline{\mathbf{F_{21}}}$
- Hence $\ddot{\underline{\mathbf{r}}} = \underline{\mathbf{F_{21}}} \left(\frac{1}{m_1} + \frac{1}{m_2} \right)$
- Define $\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2} \quad \rightarrow \quad \mu = \frac{m_1 m_2}{m_1 + m_2}$

  μ is the *reduced mass of the system*

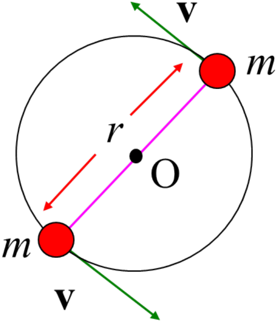
- Hence $\mu \, \ddot{\underline{\mathbf{r}}} = \underline{\mathbf{F_{21}}}$

  and $\mu \, \ddot{\underline{\mathbf{r}}} = -\frac{G m_1 m_2}{|\underline{\mathbf{r_2}} - \underline{\mathbf{r_1}}|^2} \, \hat{\underline{\mathbf{r}}} = -\frac{G \mu (m_1 + m_2)}{|\underline{\mathbf{r_2}} - \underline{\mathbf{r_1}}|^2} \, \hat{\underline{\mathbf{r}}}$

- Therefore Newton's Second Law for mutual motion can be re-written in terms of the position of the second body with respect to the first. The second body has the reduced mass which orbits round the first body with an effective mass equal to the sum of the two masses.

19.3.1 Example: binary star

A binary star consists of two stars bound together by gravity moving in roughly opposite directions along a nearly circular orbit. The period of revolution of the starts about their centre of mass is 14.4 days and the speed of each component is 220 km s^{-1}. Find the distance between the two stars and their masses.



- For single star : $v = (\frac{r}{2})\,\omega = \frac{r}{2}\frac{2\pi}{T}$
- $r = \frac{vT}{\pi} = 8.7 \times 10^{10}$ m
- Mutual motion : $\mu\,(\ddot{r} - r\,\dot{\theta}^2)\underline{\hat{\mathbf{r}}} = -\frac{Gm_1 m_2}{r^2}\underline{\hat{\mathbf{r}}}$
- For circular motion :

  $\ddot{r} = \dot{r} = 0\ ,\quad r\,\dot{\theta}^2 = \text{constant} = r\omega^2$
- Equating forces :

  $r\mu\omega^2 = \frac{Gm_1 m_2}{r^2} = \frac{G\mu(m_1+m_2)}{r^2}$
- $(m_1 + m_2) = \frac{r^3\omega^2}{G}$; $m_1 = m_2$ (symmetry)
- $m_1 = m_2 = 1.25 \times 10^{32}$ kg