Vectors and Matrices Problems

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Problem Set 4: Matrices Part II

1. Give geometric interpretations of cases where the solution of a set of three linear equations in three variables (a) is unique, (b) does not exist, (c) is a line, and (d) is a plane.

2. Whilst you might find this question slightly repetitive, it introduces different methods that can be used to solve simultaneous equations.
Solve the following equations for x, y and z by:

a) Calculating the matrix inverse
b) Using Cramer's method
c) Using row reduction.

$$\begin{aligned} x + 2y + 3z &= 2 \\ 3x + 4y + 5z &= 4 \\ x + 3y + 4z &= 6 \end{aligned}$$

3. For what values of λ does the system of equations:

$$\begin{aligned} (2-\lambda)x + y + 2z &= 0 \\ x + (4-\lambda)y - z &= 0 \\ 2x - y + (2-\lambda)z &= 0 \end{aligned}$$

have a non trivial solution? Show that the three allowed solutions can be geometrically represented by the lines: (1) $x = z$; (2) $z = -x = \frac{(2+\sqrt{6})y}{2}$; and (3) $z = -x = \frac{(2-\sqrt{6})y}{2}$.

4. Show that the following equations have solutions only if $\eta = 1$ or 2, and find the solutions in these cases:

$$\begin{aligned} x + y + z &= 1 \\ x + 2y + 4z &= \eta \\ x + 4y + 10z &= \eta^2 \end{aligned}$$

5. Find the condition(s) on α such that the simultaneous equations

$$\begin{aligned} x + \alpha y &= 1 \\ x - y + 3z &= -1 \\ 2x - 2y + \alpha z &= -2 \end{aligned}$$

have (a) exactly one solution, (b) no solutions, or (c) an infinite number of solutions. Give the form of the solutions for the case (c).

6. Find the value for α such that the three planes:

$$\begin{aligned} 3x + 2y - z &= 10 \\ 5x - y - 4z &= 17 \\ x + 5y + \alpha z &= \beta \end{aligned}$$

do not intersect in a single point. Show that for this value of α there is no point common to all these planes unless $\beta = 3$. For the case of the three planes having a common line of intersection, find its equation in cartesian form.

7. Find the eigenvalues and a set of normalized eigenvectors of the matrix:

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

Verify that its eigenvectors are mutually orthogonal.

8. An ellipse, defined by the equation $x^2 + 3y^2 - 2xy = 1$, may be rewritten in matrix form as $\mathbf{x}^T\,\mathbf{A}\mathbf{x} = 1$ where $\mathbf{x}$ is the column vector $\begin{pmatrix} x \\ y \end{pmatrix}$ and $\mathbf{A}$ is a symmetric matrix. Determine $\mathbf{A}$.

9. Find the normalized eigenvectors of the Hermitian matrix

$$H = \begin{pmatrix} 10 & 3i \\ -3i & 2 \end{pmatrix}.$$

Hence construct a unitary matrix U such that $U^\dagger H U = \Lambda$ where Λ is the real diagonal matrix

$$\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & 11 \end{pmatrix}$$

which has the eigenvalues of H as its diagonal elements.

[Hint: Avoid a common mistake. Remember that the normalization condition for a matrix A which has imaginary elements is $A^\dagger . A = 1$ (as opposed to $A^T . A = 1$).]

10. Show that the quadratic surface

$$5x^2 + 11y^2 + 5z^2 - 10yz + 2xz - 10xy = 4$$

is an ellipsoid with semi-axes of lengths $2, 1$ and 0.5. Find the direction of its longest axis.

[Hint: You might find this question quite tricky, but nevertheless it is very instructive. First write the left hand side of the equation in matrix form $\mathbf{x}^T A\mathbf{x}$ and then calculate the eigenvectors and eigenvalues of A (trust me, these are integer values). This will then allow a change of basis from which the answer just drops out. If you need a good reference for this question, see Riley, Hobson and Bence.]