Vectors and Matrices Problems

Prof Neville Harnew, MT2012

Problem Set 3: Matrices Part I

1. Simple introduction to some properties of matrices. Evaluate/show the following:

a) If

$$A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & 2 \end{pmatrix} \text{ and } B = \begin{pmatrix} -1 & 5 & -2 \\ 2 & 2 & -1 \end{pmatrix},$$

find $A + B$ and $2A - 3B$.

b) In general, matrices are non-commutative ($AB \neq BA$). Demonstrate this for

$$A = \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}.$$

c) However, show that a matrix of the form

$$A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

does have a commutative product.

d) Matric multiplication is associative, i.e. $(AB)C = A(BC)$. Show this for

$$A = \begin{pmatrix} 2 & 1 & -1 \\ 3 & 1 & 2 \end{pmatrix}, B = \begin{pmatrix} 1 & 1 \\ 2 & 0 \\ 3 & -1 \end{pmatrix}, C = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

e) If A is the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

calculate A^2 and A^3.

f) If A, B are square $n \times n$ matrices, then the traces of the products are given by Tr$\{AB\}$ =Tr$\{BA\}$. Show this for

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 4 & 1 \\ 3 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 1 & 2 \\ 1 & 1 & 0 \\ -1 & 2 & 1 \end{pmatrix}.$$

g) If A, B are arbitrary $m \times n$ matrices, show that $(A + B)^T = A^T + B^T$, where T represents the transpose.

2. Find the inverse of $A.B$ with

$$A = \begin{pmatrix} -3 & -5 \\ 5 & 9 \end{pmatrix}, \quad B = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}.$$

3. Find the two-dimensional matrix A which has the properties:

$$A \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and

$$A^{-1} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

4. For which values of α is the matrix

$$A = \begin{pmatrix} 1 & -3 & 2 \\ -\alpha & -1 & 2 \\ 3 & \alpha & -4 \end{pmatrix}$$

not invertible (i.e. the inverse doesn't exist)?

5. Find a general 2×2 matrix A such that

$$A^2 = -I = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

What are the possible sets of elements of A if the matrix is (i) diagonal and (ii) anti-diagonal?

6. Let A be a square matrix

a) If $A^2 = 0$ show that $I - A$ is invertible.

b) If $A^3 = 0$ show that $I - A$ is invertible.

c) If $A^n = 0$ for some positive integer n, show that $I - A$ is invertible.

d) If $A^2 + 2A + I = 0$, show that A is invertible.

7. Calculate the determinants of the matrices

$$(a)\ A = \begin{pmatrix} 0 & -i & i \\ i & 0 & -i \\ -i & i & 0 \end{pmatrix}, \quad (b)\ B = \frac{1}{\sqrt{8}} \begin{pmatrix} \sqrt{3} & -\sqrt{2} & -\sqrt{3} \\ 1 & \sqrt{6} & -1 \\ 2 & 0 & 2 \end{pmatrix}.$$

Are the matrices (i) real, (ii) diagonal, (iii) symmetric, (iv) antisymmetric, (v) singular, (vi) orthogonal, (vii) Hermitian, (viii) anti-Hermitian, (ix) unitary, (x) normal?

8. A vector $\mathbf{y}$ is related to vector $\mathbf{x}$ by the equation

$$\mathbf{y} = K\mathbf{x}$$

where K is a matrix. For the case when $\mathbf{x}$ and $\mathbf{y}$ are both 3−dimensional vectors, it is found that $\mathbf{y} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ when $\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$, $\mathbf{y} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ when $\mathbf{x} = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$, and $\mathbf{y} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ when $\mathbf{x} = \begin{pmatrix} 1 \\ -2 \\ -5 \end{pmatrix}$.

Write down the inverse matrix K^{-1} and then calculate the matrix K.

9. A is a non-singular 3×3 matrix and $B = 2A^{-1}$. Calculate $\mathrm{Tr}\{AB\}$ and $\det(A)\det(B)$.

10. Demonstrate that the determinant of a 3×3 matrix does not change if one of the matrix rows is multiplied by a constant k and added to another row so that the new matrix has two rows unchanged and one row replaced with the new elements. This property is important in the solution of systems of linear equations.

11. Prove the following results involving Hermitian matrices:

a) If A is Hermitian and U is unitary then $U^{-1}AU$ is Hermitian.

b) If A is anti-Hermitian then iA is Hermitian.

c) The product of two Hermitian matrices A and B is Hermitian if and only if A and B commute.

12. The points $\mathbf{A} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\mathbf{A}' = \begin{pmatrix} x' \\ y' \end{pmatrix}$ are related by the equation

$$\mathbf{A}' = R\mathbf{A}$$

where R is the (2×2) matrix, $R = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$. By expressing $x^2 + y^2$ and $x'^2 + y'^2$ in terms of $\mathbf{A}$ and $\mathbf{A}'$ respectively show that $x^2 + y^2 = x'^2 + y'^2$ and hence find the inverse, R^{-1}, of R.

13. A linear homogeneous transformation that moves a point $\mathbf{r} = (x, y, z)$ to a new point $\mathbf{r}' = (x', y', z')$ can be represented by a matrix M such that $M\mathbf{r} = \mathbf{r}'$. Determine the 3×3 matrices for the following situations:

(a) A counter-clockwise rotation by an angle α about the z-axis.

(b) A counter-clockwise rotation by an angle β about the x-axis.

(c) Counter-clockwise rotations, first by an angle β about the x-axis and then by an angle α about the z-axis.

For the special case of part (c) where $\alpha = \beta = \pi/2$, determine the unit vector $\mathbf{v}$ such that the effect of the transformation represented by matrix M is to leave $\mathbf{v}$ unaltered.