Classical Mechanics

LECTURE 9: INELASTIC COLLISIONS

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MT 2016

OUTLINE : 9. INELASTIC COLLISIONS

9.1 Examples of 2D elastic collisions

9.1.1 Example 1: Equal masses, target at rest

9.1.2 Example 2: Elastic collision, $m_2 = 2m_1$, $\theta_1 = 30^\circ$

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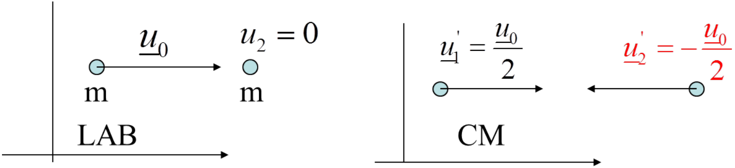
9.3.1 Kinetic energy in the CM : alternative treatment

9.3.2 Coefficient of restitution in the CM

9.3.3 Example of inelastic process

9.1.1 Example 1: Equal masses, target at rest

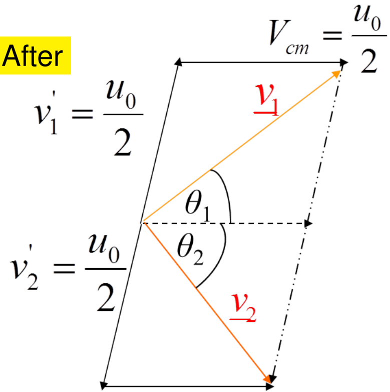
Before



Magnitude of velocities:

- $v_{CM} = \frac{m_1 u_1 + m_2 u_2}{m_1 + m_2} = \frac{u_0}{2}$
- $u_1' = u_0 - v_{CM} = \frac{u_0}{2}$
- $u_2' = -v_{CM} = -\frac{u_0}{2}$
- $|v_1'| = |u_1'| = \frac{u_0}{2}$
- $|v_2'| = |u_2'| = \frac{u_0}{2}$

After



Relationships between angles and speeds

Angles:

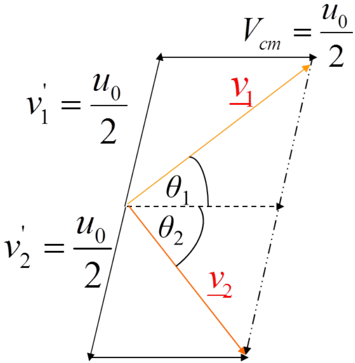
- Cosine rule:

  $(\frac{u_0}{2})^2 = (\frac{u_0}{2})^2 + v_1^2 - 2v_1\frac{u_0}{2}\cos\theta_1$
- $v_1 u_0 \cos\theta_1 = v_1^2$
- $\cos\theta_1 = \frac{v_1}{u_0}$ as before
- $\cos\theta_2 = \frac{v_2}{u_0}$

Opening angle:

- Cosine rule:

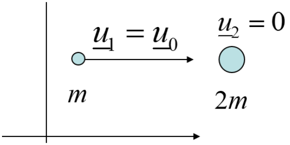
  $u_0^2 = v_1^2 + v_2^2 - 2v_1 v_2\cos(\theta_1 + \theta_2)$
- But $u_0^2 = v_1^2 + v_2^2$ (conservation of energy)
- $\cos(\theta_1 + \theta_2) = 0 \quad \rightarrow \quad \theta_1 + \theta_2 = \frac{\pi}{2}$



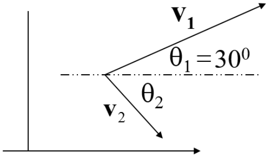
NB: Lines joining opposite corners of rhombus cross at 90°

9.1.2 Example 2: Elastic collision, $m_2 = 2m_1$, $\theta_1 = 30^\circ$

Find the velocities v_1 and v_2 and the angle θ_2



LAB before



LAB after

Magnitude of velocities:

- $v_{CM} = \frac{m_1 u_1 + m_2 u_2}{m_1 + m_2} = \frac{u_0}{3}$
- $u_1' = u_0 - v_{CM} = \frac{2u_0}{3}$
- $u_2' = -v_{CM} = -\frac{u_0}{3}$
- $|v_1'| = |u_1'| = \frac{2u_0}{3}$
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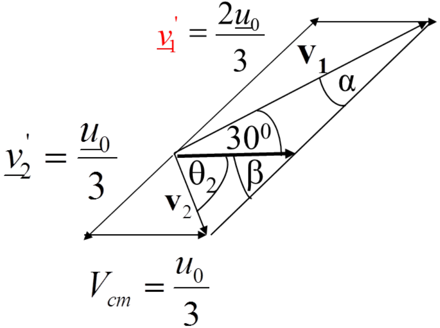


Relationships between angles and speeds

- Sine rule:

  $(\sin 30/\frac{2u_0}{3}) = (\sin\alpha/\frac{u_0}{3})$

  $\rightarrow\ \sin\alpha = \frac{1}{4}\ \rightarrow\ \alpha = 14.5^\circ$
- $\beta = 30 + \alpha = 44.5^\circ$
- $\sin 30/\frac{2u_0}{3} = \sin(180 - 44.5)/v_1$

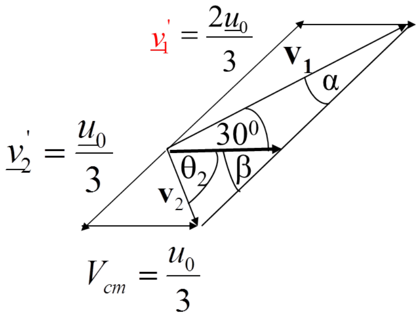
  $\rightarrow\ v_1 = 0.93u_0$
- Cosine rule:

  $v_2^2 = (\frac{u_0}{3})^2 + (\frac{u_0}{3})^2 - 2(\frac{u_0}{3})^2\cos\beta$

  $\rightarrow\ v_2 = 0.25u_0$
- Sine rule:

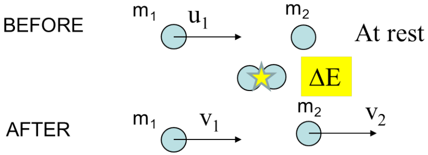
  $(\sin 44.5/v_2) = (\sin\theta_2/\frac{u_0}{3})$

  $\rightarrow\ \theta_2 = 68.0^\circ$



9.2 Inelastic collisions in the Lab frame in 1D ($u_2 = 0$)

An *inelastic* collision is where energy is lost (or there is internal excitation).



- Take m_2 at rest & in 1D. Momentum : $m_1 u_1 = m_1 v_1 + m_2 v_2$ (1)
- Energy : $\frac{1}{2} m_1 u_1^2 = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 + \Delta E$ (2)
- Square Equ.(1) and subtract $2m_1 \times$ Equ.(2)

  $\rightarrow \quad m_2(m_2 - m_1)v_2^2 + 2m_1 m_2 v_1 v_2 - 2m_1 \Delta E = 0$
- Substitute for $m_1 v_1$ from Equ.1 to get quadratic in v_2

  $\rightarrow \quad m_2(m_2 + m_1)v_2^2 - 2m_1 m_2 u_1 v_2 + 2m_1 \Delta E = 0$
- Solve, taking consistent solutions with elastic case ($\Delta E = 0$)

$$\rightarrow \quad v_2 = \frac{2m_1 m_2 u_1 + \sqrt{4m_1^2 m_2^2 u_1^2 - 8m_1 m_2 (m_1 + m_2)\Delta E}}{2m_2(m_1 + m_2)} \qquad (3)$$

$$\rightarrow \quad v_1 = \frac{2m_1^2 u_1 - \sqrt{4m_1^2 m_2^2 u_1^2 - 8m_1 m_2 (m_1 + m_2)\Delta E}}{2m_1(m_1 + m_2)} \qquad (4)$$

1D inelastic collisions viewed in the Lab frame ($u_2 = 0$)

- We see from Equ. (3) & (4) there is a *limiting case*: $4m_1^2 m_2^2 u_1^2 - 8m_1 m_2(m_1 + m_2)\Delta E \geq 0$
- i.e. $\Delta E \leq \frac{m_1 m_2 u_1^2}{2(m_1+m_2)}$
- This corresponds to the two bodies sticking together in a single object of mass $(m_1 + m_2) \quad \rightarrow \quad v_1 = v_2$
- From momentum cons. $\quad m_1 u_1 = m_1 v_1 + m_2 v_2$

  if $v_1 = v_2 = v$, then $v = \frac{m_1 u_1}{(m_1+m_2)}$ (the CM velocity)

For equal mass $m_1 = m_2$ $\quad v_2\ ,\ v_1 = \frac{u_1}{2}\left[1 \pm \sqrt{1 - \frac{4\Delta E}{mu_1^2}}\right]$

9.2.1 Coefficient of restitution

General definition : $e = \frac{|\underline{\mathbf{v}}_2 - \underline{\mathbf{v}}_1|}{|\underline{\mathbf{u}}_1 - \underline{\mathbf{u}}_2|} = \frac{\text{Speed of relative separation}}{\text{Speed of relative approach}}$

- From Equ.(3) & (4) previously

$$v_2 - v_1 = \frac{2m_1 m_2 u_1 + \sqrt{4m_1^2 m_2^2 u_1^2 - 8m_1 m_2 (m_1+m_2)\Delta E}}{2m_2(m_1+m_2)} - \frac{2m_1^2 u_1 - \sqrt{4m_1^2 m_2^2 u_1^2 - 8m_1 m_2 (m_1+m_2)\Delta E}}{2m_1(m_1+m_2)}$$

- Factorizing, then simplifying, then dividing by u_1 gives

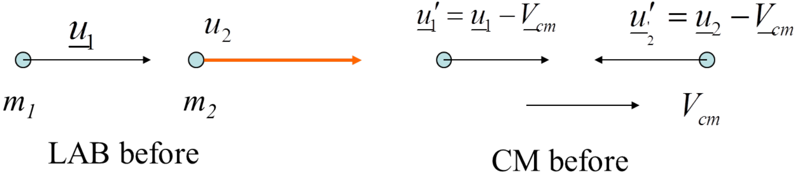
$$e = \sqrt{1 - \frac{2(m_1+m_2)\Delta E}{m_1 m_2 u_1^2}} = \sqrt{1 - \frac{\Delta E}{T'}}$$

where $T' = \frac{1}{2}\mu u_1^2$ with $\mu = \frac{m_1 m_2}{m_1+m_2}$ (the *reduced mass*)

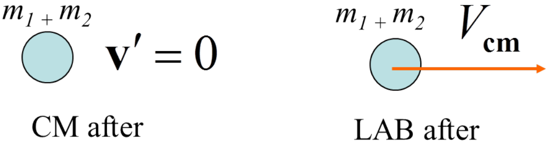
- We see later that T' is the initial energy in the CM frame, hence e is related to the fractional energy loss in this frame
- $e = 1$ completely elastic; $e = 0$ perfectly inelastic, in general $0 < e < 1$

9.3 Inelastic collisions viewed in the CM frame

Case of perfectly inelastic collision ($e = 0$)



After collision, total mass $(m_1 + m_2)$ is at rest in CM:

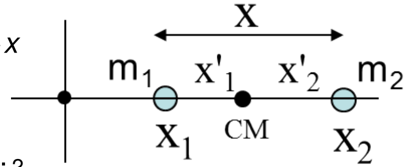


- KE in CM: $T_{CM} = T_{LAB} - \frac{1}{2}(m_1 + m_2)v_{CM}^2$
- Differentiate: Loss in KE $\Delta T_{CM} = \Delta T_{LAB}$ (obvious)
- Max. energy that can be lost = $T_{CM} =$
  $= \frac{1}{2}m_1 u_1^2 + \frac{1}{2}m_2 u_2^2 - \frac{1}{2}(m_1 + m_2)v_{CM}^2$

9.3.1 Kinetic energy in the CM : alternative treatment

Revisit kinetic energy in the CM frame: $T_{Lab} = T_{CM} + \frac{1}{2} M v_{CM}^2$

- $T_{CM} = \frac{1}{2} m_1 u_1'^2 + \frac{1}{2} m_2 u_2'^2$
- $x_1' = -\frac{m_2}{m_1+m_2} x = -\frac{m_2}{M} x \ , \ x_2' = \frac{m_1}{M} x$
- $u_1' = -\frac{m_2}{M} \dot{x} \ , \ u_2' = \frac{m_1}{M} \dot{x}$
- $T_{CM} = \frac{1}{2} \left(m_1 (-\frac{m_2}{M})^2 + m_2 (\frac{m_1}{M})^2 \right) \dot{x}^2$
- $T_{CM} = \frac{1}{2} \frac{m_1 m_2}{M^2} (m_2 + m_1) \dot{x}^2 = \frac{1}{2} \frac{m_1 m_2}{M} \dot{x}^2$
- Also $\dot{x} = \dot{x}_2 - \dot{x}_1 = u_2' - u_1'$



$$T_{CM} = \frac{1}{2} \frac{m_1 m_2}{M} \dot{x}^2 = \frac{1}{2} \mu \dot{x}^2 = \frac{1}{2} \mu (u_1' - u_2')^2 = \frac{1}{2} \mu (u_1 - u_2)^2$$

These expressions give the CM kinetic energy in terms of the relative velocities in the CM & Lab and the reduced mass μ

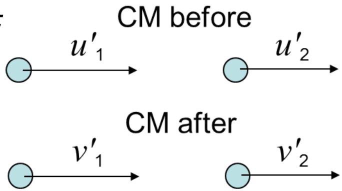
9.3.2 Coefficient of restitution in the CM

- Initial KE in the CM : $T^i_{CM} = \frac{1}{2}\mu(u'_1 - u'_2)^2$
- Final KE in the CM : $T^f_{CM} = \frac{1}{2}\mu(v'_1 - v'_2)^2$
- Conservation of energy : $T^i_{CM} = T^f_{CM} + \Delta E$

$$\rightarrow \quad \frac{1}{2}\mu(u'_1 - u'_2)^2 = \frac{1}{2}\mu(v'_1 - v'_2)^2 + \Delta E$$

$$\rightarrow \quad \left(\frac{v'_1 - v'_2}{u'_1 - u'_2}\right)^2 = 1 - \frac{\Delta E}{T^i_{CM}}$$

$$\rightarrow \quad \left(\frac{v'_1 - v'_2}{u'_1 - u'_2}\right) = \pm\sqrt{1 - \frac{\Delta E}{T^i_{CM}}}$$



Same expression as before with $T' = T^i_{CM}$

Coefficient of restitution

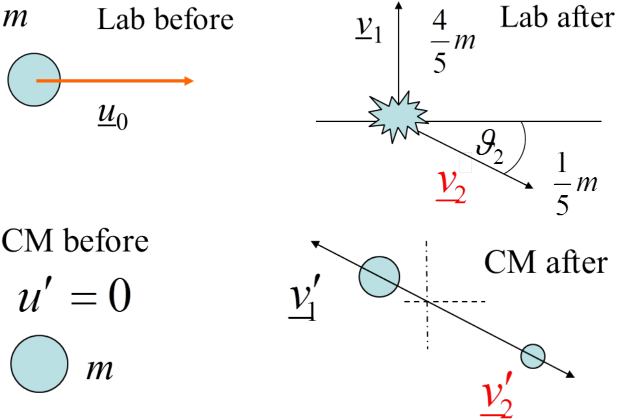
$$e = \frac{|\mathbf{v}'_2 - \mathbf{v}'_1|}{|\mathbf{u}'_1 - \mathbf{u}'_2|}_{CM} = \frac{|\mathbf{v}_2 - \mathbf{v}_1|}{|\mathbf{u}_1 - \mathbf{u}_2|}_{LAB} = \sqrt{1 - \frac{\Delta E}{T^i_{CM}}}$$

ONLY in CM frame can ALL the KE be used to create ΔE

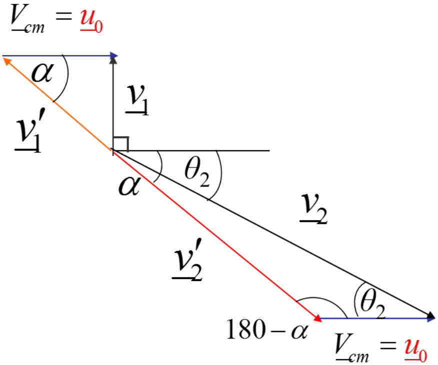
$\rightarrow$ For $e = 0$ the two particles coalesce and are at rest in CM

9.3.3 Example of inelastic process

A calcium nucleus (A=20), mass m, travels with velocity u_0 in the Lab. It decays into a sulphur nucleus (A=16), mass $\frac{4}{5}m$, and an α-particle (A=4), mass $\frac{1}{5}m$. Energy ΔT is released as KE in the calcium rest frame (CM). A counter in the Lab detects the sulphur nucleus at 90° to the line of travel. What is the speed and angle of the α-particle in the Lab?



- Energy ΔT is released as KE in the CM. $v_{CM} = u_0$
- Momentum in CM:
  $\frac{4}{5} m v_1' - \frac{1}{5} m v_2' = 0$
  $\rightarrow \quad v_2' = 4 v_1'$
- Energy: $\Delta T = \frac{1}{2}(\frac{4}{5} m) {v_1'}^2 + \frac{1}{2}(\frac{1}{5} m) 16 {v_1'}^2 = 2m {v_1'}^2$
  $\rightarrow \quad v_1' = [\frac{\Delta T}{2m}]^{\frac{1}{2}}$
  $\rightarrow \quad v_2' = [\frac{8 \Delta T}{m}]^{\frac{1}{2}}$
- Transform to Lab by boosting by $v_{CM} (= u_0)$



- $\cos \alpha = \frac{u_0}{v_1'} = [\frac{2 m u_0^2}{\Delta T}]^{\frac{1}{2}}$
- Cosine rule: $v_2^2 = {v_2'}^2 + {u_0}^2 + 2 v_2' u_0 \cos \alpha$
- Sine rule: $\frac{\sin \theta_2}{v_2'} = \frac{\sin \alpha}{v_2}$

Solve for v_2 , θ_2