LECTURE 9:

SPECIAL MATRICES AND MATRIX OPERATIONS

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Outline: 9. SPECIAL MATRICES AND MATRIX OPERATIONS

9.1 The diagonal matrix

A is diagonal if $A_{ij} = 0$ for $i \neq j$.

- i.e. the matrix has only elements on the diagonal which are different from zero.
- Obviously this is defined only if $n = m$ (a square matrix).
- Example:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} \tag{1}$$

- An anti-diagonal matrix - example:

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 3 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{pmatrix} \tag{2}$$

9.2 The identity (unit) and null (zero) matrices

The unit matrix:

- A diagonal matrix I with all diagonal elements = 1.

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \tag{3}$$

- This has the property $AI = IA = A$.

The null matrix:

- A matrix 0 with all elements equal to zero:
  - $A\,0 = 0$
  - $A + 0 = A$

9.3 Transpose of a matrix

- The transpose of a matrix A, size $m \times n$, is a matrix B of size $n \times m$ with the rows and columns of A interchanged.
- $B = A^T \quad \Rightarrow \quad B_{ji} = A_{ij}$
- Example

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}, \quad A^T = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix} \tag{4}$$

9.3.1 Notes on matrix transpose

- A *symmetric* matrix is a square matrix with $A = A^T$

Example : $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 6 & 7 \\ 3 & 6 & 8 & 9 \\ 4 & 7 & 9 & 0 \end{pmatrix}$

- If $A = -A^T$ the matrix is antisymmetric.

Example : $\begin{pmatrix} 0 & 2 & 3 & 4 \\ -2 & 0 & 6 & 7 \\ -3 & -6 & 0 & 9 \\ -4 & -7 & -9 & 0 \end{pmatrix}$

- Also $(AB)^T = B^T A^T$ (note that the order of A and B is reversed).

  Proof

  - Consider a general element of the matrix product:
  - $$\begin{aligned}(AB)^T_{ij} &= (AB)_{ji} \\ &= \sum_k A_{jk} B_{ki} \\ &= \sum_k (A^T)_{kj} (B^T)_{ik} \\ &= \sum_k (B^T)_{ik} (A^T)_{kj} \\ &= (B^T A^T)_{ij} \qquad \text{QED.}\end{aligned}$$

- It follows $(ABC...Z)^T = Z^T ... C^T B^T A^T$

9.4 The complex and Hermitian conjugates

- To get the *complex conjugate* of a matrix A - take the complex conjugate of each element: $(A^*)_{ij} = (A_{ij})^*$
- If a matrix is real: then $A^* = A$.
- The *Hermitian conjugate* of a matrix is the complex conjugate of its transpose:
  $A^\dagger = (A^T)^* = (A^*)^T$
  (Remember this was used for the inner product earlier).

To note:

- In analogy to the property of the transpose, $(ABC...Z)^\dagger = Z^\dagger ... C^\dagger B^\dagger A^\dagger$.
- A complex matrix with $A = A^\dagger$ is *Hermitian*.
- If $A = -A^\dagger$, the matrix is anti-Hermitian.

9.4.1 Example

- Take the Hermitian conjugate of the matrix A

$$A = \begin{pmatrix} i & 2+i \\ -2+i & 3i \end{pmatrix} \tag{7}$$

$$\text{hence } A^* = \begin{pmatrix} -i & 2-i \\ -2-i & -3i \end{pmatrix} \tag{8}$$

$$\text{and } A^\dagger = \begin{pmatrix} -i & -2-i \\ 2-i & -3i \end{pmatrix} \tag{9}$$

- In this case, since $A^\dagger = -A$, the matrix is anti-Hermitian.

9.5 The inverse of a matrix

- For a matrix A, the *inverse* of the matrix A^{-1} is such that: $AA^{-1} = A^{-1}A = I$.
  This obviously applies only to square matrices.
- Not all square matrices have inverses. If a matrix has an inverse then the matrix is $invertable$ or $non-singular$. Otherwise its $singular$.

Division of matrices

- There is no matrix division !
- However: if we want to find B where $A.B = C$ (all square matrices of the same dimension), then multiply from the left by A^{-1}:

$$A^{-1}A.B = B = A^{-1}C.$$

- How to invert a matrix comes later.

9.5.1 Inverse matrix: a few useful properties

i) $(A^{-1})^{-1} = A$ (follows directly from $AA^{-1} = A^{-1}A = I$).

ii) $(A^T)^{-1} = (A^{-1})^T$

iii) Similarly $(A^\dagger)^{-1} = (A^{-1})^\dagger$

iv) $(AB)^{-1} = B^{-1}A^{-1}$

Proof:

- Start from $(AB)(AB)^{-1} = I$
- Multiply both sides from the left by A^{-1},
  $A^{-1}(AB)(AB)^{-1} = A^{-1}$
- Hence $B(AB)^{-1} = A^{-1}$
- Multiply both sides from the left by B^{-1},
  $B^{-1}B(AB)^{-1} = (AB)^{-1} = B^{-1}A^{-1}$ as required.

9.6 Orthogonal, unitary & normal matrices

- A real matrix with $A^T = A^{-1}$ is *orthogonal*,
- A matrix with $A^\dagger = A^{-1}$ is *unitary*,
- A matrix with $AA^\dagger = A^\dagger A$ is *normal*, commutes with its Hermitian congugate.

Commuting matrices

- Note that for matrices A and B to commute: $AB - BA = 0$
  Multiply from the left by A^{-1} $\rightarrow$ $B - A^{-1}BA = 0$

  The condition is: $B = A^{-1}BA$

9.7 Trace of an $n \times n$ matrix

- Defined as the sum of diagonal elements (the matrix must be square):

$$\text{Tr}\, A = A_{11} + A_{22} + \cdots + A_{nn} = \sum_{i=1}^{n} A_{ii}$$

- Can easily show
  - $\text{Tr}(A \pm B) = \text{Tr}A \pm \text{Tr}B$
  - $\text{Tr}(ABC) = \text{Tr}(CAB) = \text{Tr}(BCA)$ (cyclic permutations).

9.8 Functions of a matrix

- Powers of a matrix
  $A^2 = A.A; \quad A^3 = A.A.A; \quad A^n = A.A.A \, (n \text{ times})$

- Exponential of a matrix
  $\exp A = 1 + A + \frac{A.A}{2!} + \frac{A.A.A}{3!} + \cdots$
  $= 1 + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{A^n}{n!}$
  where $1 \equiv I \quad \rightarrow$ the unit (or identity) matrix

- Sine of a matrix
  $\sin A = A - \frac{A^3}{3!} + \frac{A^5}{5!} + \cdots$

- Can similarly expand anything that has a series.