LECTURE 8: BASIC MATRIX ALGEBRA

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Outline: 8. BASIC MATRIX ALGEBRA

8.1 Addition of matrices

- Matrix summation follows the properties of linear algebra

$$C = A + B$$

$$\text{Hence} \quad C_{ij} = (A + B)_{ij} = A_{ij} + B_{ij}$$

- Writing this out:

$$C = A + B = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{pmatrix} + \begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1n} \\ B_{21} & B_{22} & \cdots & B_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ B_{m1} & B_{m2} & \cdots & B_{mn} \end{pmatrix} \tag{1}$$

$$= \begin{pmatrix} A_{11} + B_{11} & A_{12} + B_{12} & \cdots & A_{1n} + B_{1n} \\ A_{21} + B_{21} & A_{22} + B_{21} & \cdots & A_{2n} + B_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ A_{m1} + B_{m1} & A_{m2} + B_{m1} & \cdots & A_{mn} + B_{mn} \end{pmatrix} \tag{2}$$

- For this to have any meaning, both matrices must have the same dimensions (in this case m $\times$ n).

8.1.1 Properties and an example

- It follows obviously (and also from the rules of linear operators) that $C = A + B = B + A$ (commutative).
- The difference of two matrices follows in an obvious way: $C = A - B$ has elements $C_{ij} = A_{ij} - B_{ij}$.

Example

$$C = A+B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} + \begin{pmatrix} 7 & 8 & 9 \\ -2 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 8 & 10 & 12 \\ 2 & 4 & 6 \end{pmatrix} \tag{3}$$

8.2 Multiplication by a scalar

- This has the property $\lambda A_{ij} = (\lambda A)_{ij}$

$$\lambda \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{pmatrix} = \begin{pmatrix} \lambda A_{11} & \lambda A_{12} & \cdots & \lambda A_{1n} \\ \lambda A_{21} & \lambda A_{22} & \cdots & \lambda A_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \lambda A_{m1} & \lambda A_{m2} & \cdots & \lambda A_{mn} \end{pmatrix} \tag{4}$$

- This operation is distributative and associative:
  - $\lambda(A + B) = \lambda A + \lambda B$
  - $(\lambda + \mu)A = \lambda A + \mu A$
  - $(\lambda\mu)A = \lambda(\mu A)$

8.2.1 Example

If $$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} \tag{5}$$

Then evaluate D = A - 2 B + 3 C

$$\rightarrow \mathrm{D} = \begin{pmatrix} 1-2+15 & 2+2+18 \\ 3+2+21 & 4-2+24 \end{pmatrix} \tag{6}$$

$$= \begin{pmatrix} 14 & 22 \\ 26 & 26 \end{pmatrix} \tag{7}$$

8.3 Multiplication of matrices

- In matrix multiplication: an m $\times$ n matrix A can be multiplied by an n $\times$ p matrix B giving an m $\times$ p matrix C.

$$C = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{pmatrix} \cdot \begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1p} \\ B_{21} & B_{22} & \cdots & B_{2p} \\ \cdots & \cdots & \cdots & \cdots \\ B_{n1} & B_{n2} & \cdots & B_{np} \end{pmatrix} \tag{8}$$

$$= \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1p} \\ C_{21} & C_{22} & \cdots & C_{2p} \\ \cdots & \cdots & \cdots & \cdots \\ C_{m1} & C_{m2} & \cdots & C_{mp} \end{pmatrix} \tag{9}$$

where $C_{ij} = A_{i1}B_{1j} + A_{i2}B_{2j} + \cdots + A_{in}B_{nj}$
i.e. $C_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj}$ for all $i = 1$ to m and all $j = 1$ to p.

- Note that the number of columns in A must equal the number of rows in B.

8.3.1 Example 1

- Transformation of a column vector using matrix multiplication:

$$\begin{pmatrix} y_1 \\ y_2 \\ . \\ y_m \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ . \\ x_n \end{pmatrix} \tag{10}$$

- The elements of y are given by

$$\begin{pmatrix} y_1 \\ y_2 \\ . \\ y_m \end{pmatrix} = \begin{pmatrix} A_{11}x_1 + A_{12}x_2 + \cdots + A_{1n}x_n \\ A_{21}x_1 + A_{22}x_2 + \cdots + A_{2n}x_n \\ \cdots \\ A_{m1}x_1 + A_{m2}x_2 + \cdots + A_{mn}x_n \end{pmatrix} \tag{11}$$

8.3.2 Example 2

- Multiplying a 2 × 3 matrix by a 3 × 2 matrix:

$$\begin{pmatrix} 3 & -4 & 2 \\ 1 & 5 & -2 \end{pmatrix} \times \begin{pmatrix} 2 & -1 \\ 7 & 3 \\ 5 & -4 \end{pmatrix} \tag{12}$$

$$= \begin{pmatrix} 3 \times 2 + (-4) \times 7 + 2 \times 5 & 3 \times (-1) + (-4) \times 3 + 2 \times (-4) \\ 1 \times 2 + 5 \times 7 + (-2) \times 5 & 1 \times (-1) + 5 \times 3 + (-2) \times (-4) \end{pmatrix} \tag{13}$$

$$= \begin{pmatrix} -12 & -23 \\ 27 & 22 \end{pmatrix} \tag{14}$$

8.4 Properties of matrix multiplication [1]

- Multiplication of matrices is not in general commutative (even for square matrices).

$$AB \neq BA \quad \text{since} \quad \sum_{k=1}^{n} A_{ik} B_{kj} \neq \sum_{k=1}^{n} B_{ik} A_{kj}$$

Example:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & -1 \\ 7 & 3 \end{pmatrix} \tag{15}$$

Then

$$AB = \begin{pmatrix} 2 & -1 \\ 21 & 9 \end{pmatrix}, \quad BA = \begin{pmatrix} 2 & -3 \\ 7 & 9 \end{pmatrix} \neq AB \tag{16}$$

- An aside: 2×2 matrices of the form

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \tag{17}$$

do commute with other matrices of the same form.

- Multiplication of more than two matrices is associative (provided the dimensions are correct for a meaningful product):

$$A(BC) = (AB)C$$

8.4.1 Proof of $A(BC) = (AB)C$

- Consider the elements of the matrix products. Take A, B, C as square matrices $n \times n$:
  $\rightarrow \quad (BC)_{ij} = \sum_{k=1}^{n} B_{ik} C_{kj}$
  Hence $\quad (A(BC))_{mj} = \sum_{i=1}^{n} A_{mi} (\sum_{k=1}^{n} B_{ik} C_{kj})$
- A_{mi} is just a number, take it inside 2nd summation
  Hence $\quad (A(BC))_{mj} = \sum_{i=1}^{n} \sum_{k=1}^{n} (A_{mi} B_{ik} C_{kj})$
- Change order of summation
  $(A(BC))_{mj} = \sum_{k=1}^{n} \left(\sum_{i=1}^{n} A_{mi} B_{ik} \right) C_{kj}$
  $= ((AB)C)_{mj} \qquad$ QED

Properties of matrix multiplication [2]

- Matrix multiplication is distributive over addition:

$$(A+B)C = AC + BC$$
$$C(A+B) = CA + CB$$

Example:

- Expand $(A+B)^2$ where A and B are square matrices.
  $(A+B)^2 = (A+B).(A+B) = A^2 + A.B + B.A + B^2$
- Note this is not equal to $A^2 + 2A.B + B^2$!