Classical Mechanics

LECTURE 7:

TWO-BODY ELASTIC COLLISIONS

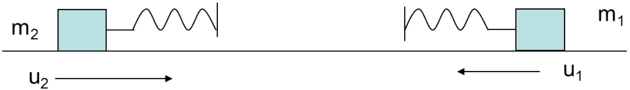
Prof. N. Harnew

University of Oxford

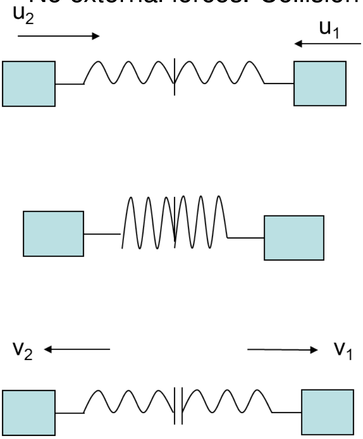
MT 2016

OUTLINE : 7. TWO-BODY ELASTIC COLLISIONS

7.1 Two-body collisions - general concepts



No external forces. Collision via massless springs or other force type.

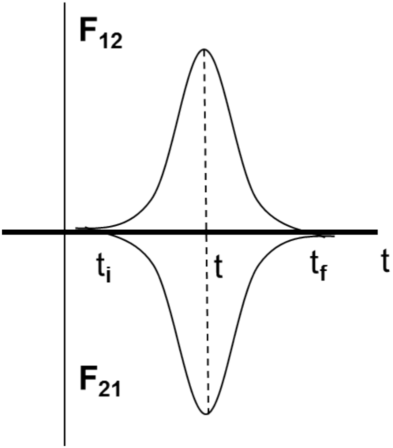


- t_i : collision starts. All energy is kinetic. $T_i = \frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2$
- t : collision peaks. Some kinetic is converted into potential (of the spring). $E = \frac{1}{2} m_1 \dot{x}_1{}^2 + \frac{1}{2} m_2 \dot{x}_2{}^2 + E_{int}$
- t_f : collision ends. All energy is kinetic again. $T_f = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$; $T_i = T_f + \Delta E$ ($\leftarrow$ *inelastic*)

7.1.1 Momentum exchange and impulse

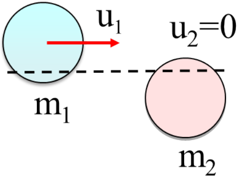
During collision: internal force causes change of momentum $\underline{\mathbf{F}} = \frac{d\underline{\mathbf{p}}}{dt}$

- At t_i : total momentum
  $\underline{\mathbf{p}} = \underline{\mathbf{p}}_1 + \underline{\mathbf{p}}_2 = m_1\underline{\mathbf{u}}_1 + m_2\underline{\mathbf{u}}_2$
- At t : $m_1 \rightarrow d\underline{\mathbf{p}}_1 = \underline{\mathbf{F}}_{12}dt$
  $m_2 \rightarrow d\underline{\mathbf{p}}_2 = \underline{\mathbf{F}}_{21}dt$
- At t_f : $m_1 \rightarrow \underline{\mathbf{v}}_1$ and $m_2 \rightarrow \underline{\mathbf{v}}_2$
- Impulse $\Delta\underline{\mathbf{p}}_1 = \underline{\mathbf{I}}_1 = \int_{t_i}^{t_f} \underline{\mathbf{F}}_{12}dt$
  $\Delta\underline{\mathbf{p}}_2 = \underline{\mathbf{I}}_2 = \int_{t_i}^{t_f} \underline{\mathbf{F}}_{21}dt$
- Since $\underline{\mathbf{F}}_{12} + \underline{\mathbf{F}}_{21} = 0$
  $\Delta\underline{\mathbf{p}}_1 + \Delta\underline{\mathbf{p}}_2 = 0$
- Momentum conserved.

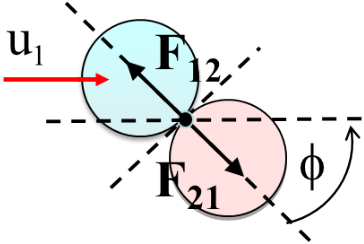


7.1.2 An off-axis collision in 2D

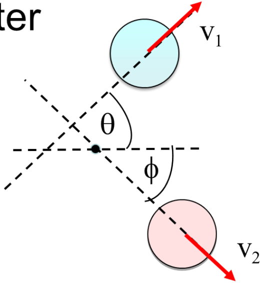
Before



During



After



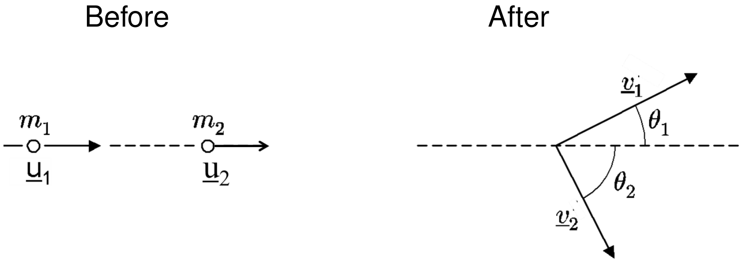
- Impulse is along line of centres

$$\Delta \underline{\mathbf{p}}_1 = \int_{t_i}^{t_f} \underline{\mathbf{F}}_{12} dt$$

$$\Delta \underline{\mathbf{p}}_2 = \int_{t_i}^{t_f} \underline{\mathbf{F}}_{21} dt$$

- $\underline{\mathbf{v}}_1 = \frac{1}{m_1} \int_{t_i}^{t_f} \underline{\mathbf{F}}_{12} dt + \underline{\mathbf{u}}_1$
- $\underline{\mathbf{v}}_2 = -\frac{1}{m_2} \int_{t_i}^{t_f} \underline{\mathbf{F}}_{12} dt$

7.2 Elastic collisions in the Lab frame

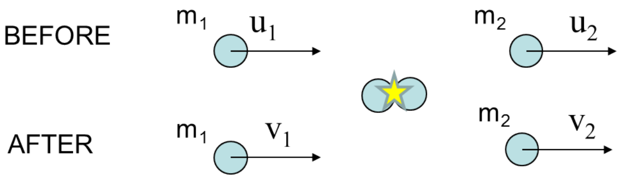


Conservation of momentum: $m_1\underline{\mathbf{u}}_1 + m_2\underline{\mathbf{u}}_2 = m_1\underline{\mathbf{v}}_1 + m_2\underline{\mathbf{v}}_2$

Conservation of energy: $\frac{1}{2}m_1\underline{\mathbf{u}}_1^2 + \frac{1}{2}m_2\underline{\mathbf{u}}_2^2 = \frac{1}{2}m_1\underline{\mathbf{v}}_1^2 + \frac{1}{2}m_2\underline{\mathbf{v}}_2^2$

[Note that the motion is in a plane, and the 2D representation can be trivially extended into 3D by rotation of the plane].

7.2.1 Elastic collisions in 1D in the Lab frame



- Momentum : $m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2$ (1)

  $\rightarrow$ $m_1(v_1 - u_1) = m_2(u_2 - v_2)$ (2)

- Energy : $\frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2 = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$

  $\rightarrow$ $m_1(v_1 - u_1)(v_1 + u_1) = m_2(u_2 - v_2)(u_2 + v_2)$ (3)

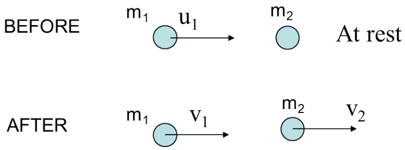
- Divide (2) & (3) :

  $\rightarrow$ $(v_1 + u_1) = (u_2 + v_2)$ $\rightarrow$ $(u_1 - u_2) = (v_2 - v_1)$ (4)

  $\rightarrow$ Relative speed before collision = Relative speed after

7.2.2 Special case in 1D where target particle is at rest

- $u_2 = 0$; From (1) & (4) :
  $m_1 u_1 = m_1 v_1 + m_2 (u_1 + v_1)$
- $v_1 = \frac{(m_1 - m_2) u_1}{m_1 + m_2}$
- Similarly :
  $m_1 u_1 = m_1 (v_2 - u_1) + m_2 v_2$
- $v_2 = \frac{2 m_1 u_1}{m_1 + m_2}$

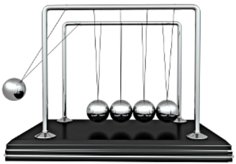


$$v_1 = \frac{m_1 - m_2}{m_1 + m_2} u_1 \quad \text{and} \quad v_2 = \frac{2 m_1}{m_1 + m_2} u_1$$

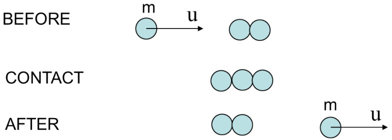
Special cases:

- $m_1 = m_2 : \rightarrow \quad v_1 = 0, \; v_2 = u_1$
  (complete transfer of momentum)
- $m_1 >> m_2$: Gives the limits $v_1 \rightarrow u_1, \; v_2 \rightarrow 2u_1$
  (m_2 has double u_1 velocity)
- $m_1 << m_2$: Gives the limits $v_1 \rightarrow -u_1, \; v_2 \rightarrow 0$
  ("brick wall" collision)

Example: Newton's cradle



Consider here just 3 balls



- If the balls are touching, the most general case is:

  Momentum after collision : $mu = mv_1 + mv_2 + mv_3$

  Energy after collision : $\frac{1}{2}mu^2 = \frac{1}{2}mv_1^2 + \frac{1}{2}mv_2^2 + \frac{1}{2}mv_3^2$

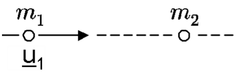
  2 equations, 3 unknowns
- The obvious solution: $v_1 = v_2 = 0, \quad v_3 = u$
- But other solution(s) possible:

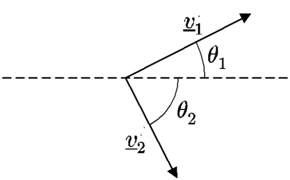
  Momentum : $mu = -\frac{1}{3}mu + \frac{2}{3}mu + \frac{2}{3}mu$

  Energy : $\frac{1}{2}mu^2 = \frac{1}{18}mu^2 + \frac{4}{18}mu^2 + \frac{4}{18}mu^2$
- So why does the simple solution always prevail?

7.2.3 Collision in 2D : equal masses, target at rest

$$m_1 = m_2 = m \, , \quad u_2 = 0$$





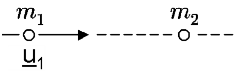
- Momentum: $m\underline{\mathbf{u}}_1 = m\underline{\mathbf{v}}_1 + m\underline{\mathbf{v}}_2 \;\rightarrow\; \underline{\mathbf{u}}_1 = \underline{\mathbf{v}}_1 + \underline{\mathbf{v}}_2$

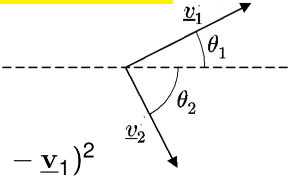
  Squaring $\rightarrow\; \underline{\mathbf{u}}_1^2 = \underline{\mathbf{v}}_1^2 + \underline{\mathbf{v}}_2^2 + 2\underline{\mathbf{v}}_1.\underline{\mathbf{v}}_2$
- Energy: $\frac{1}{2}m\underline{\mathbf{u}}_1^2 = \frac{1}{2}m\underline{\mathbf{v}}_1^2 + \frac{1}{2}m\underline{\mathbf{v}}_2^2 \;\rightarrow\; \underline{\mathbf{u}}_1^2 = \underline{\mathbf{v}}_1^2 + \underline{\mathbf{v}}_2^2$
- Hence $2\underline{\mathbf{v}}_1.\underline{\mathbf{v}}_2 = 0$

  $\rightarrow$ EITHER $\underline{\mathbf{v}}_1 = 0$ & $\underline{\mathbf{v}}_2 = \underline{\mathbf{u}}_1$ OR $(\theta_1 + \theta_2) = \frac{\pi}{2}$
- Either a head-on collision or opening angle is 90°

Relationship between speeds and angles

$$m_1 = m_2 = m\,, \quad u_2 = 0$$





- From cons. mom. $(\underline{\mathbf{v}}_2)^2 = (\underline{\mathbf{u}}_1 - \underline{\mathbf{v}}_1)^2$
  $\rightarrow \; v_2^2 = v_1^2 + u_1^2 - 2u_1 v_1 \cos\theta_1$
- Energy: $u_1^2 = v_1^2 + v_2^2 \quad \rightarrow \quad v_2^2 = u_1^2 - v_1^2$
- Equate : $2v_1^2 = 2u_1 v_1 \cos\theta_1$

$\cos\theta_1 = \frac{v_1}{u_1}$ and by symmetry $\cos\theta_2 = \frac{v_2}{u_1}$

- Note we can also do this via components of momentum :
  $\rightarrow \; u_1 = v_1 \cos\theta_1 + v_2 \cos\theta_2$ and $v_1 \sin\theta_1 = v_2 \sin\theta_2$
  $\rightarrow \; (u_1 - v_1 \cos\theta_1)^2 = v_2^2 \cos^2\theta_2$ and $v_1^2 \sin^2\theta_1 = v_2^2 \sin^2\theta_2$
  $\rightarrow$ Add : $v_2^2 = v_1^2 \sin^2\theta_1 + u_1^2 - 2u_1 v_1 \cos\theta_1 + v_1^2 \cos^2\theta_1$
  $\rightarrow$ Gives : $v_2^2 = v_1^2 + u_1^2 - 2u_1 v_1 \cos\theta_1$