LECTURE 7: INNER PRODUCTS, LINEAR OPERATORS AND INTRODUCTION TO MATRICES

Prof. N. Harnew
University of Oxford

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Outline: 7. INNER PRODUCTS, LINEAR OPERATORS AND INTRODUCTION TO MATRICES

7.1 The scalar (inner) product

First consider REAL vectors

- The inner product of two vectors $|\mathbf{a}\rangle$ and $|\mathbf{b}\rangle$ in an N dimensional vector space is denoted by $\langle\mathbf{a}|\mathbf{b}\rangle$
- The inner product of real vectors $|\mathbf{a}\rangle$ and $|\mathbf{b}\rangle$ in N dimensions (provided the basis vectors are orthogonal):

$$\langle\mathbf{a}|\mathbf{b}\rangle = \underline{\mathbf{a}}^T.\underline{\mathbf{b}} = (a_1, a_2, \ldots a_N).\begin{pmatrix} b_1 \\ b_2 \\ . \\ b_N \end{pmatrix} = \sum_{i=1}^{N} a_i b_i. \quad (1)$$

  - Here $\langle\mathbf{a}|$ is the transpose of $|\mathbf{a}\rangle$.
  - The inner product is a scalar.
  - The *norm* of $|\mathbf{a}\rangle$, i.e. $||\mathbf{a}\rangle|$, is given by $\sqrt{\langle\mathbf{a}|\mathbf{a}\rangle}$ and is a measure of (real) length : $|\underline{\mathbf{a}}| = \sqrt{\underline{\mathbf{a}}^T.\underline{\mathbf{a}}} = \sqrt{\sum_{i=1}^{N} a_i^2}$.

Other properties:

- $\langle\mathbf{a}|\mathbf{b}\rangle = \langle\mathbf{b}|\mathbf{a}\rangle$ for real vectors.
- If $|\mathbf{c}\rangle = \alpha|\mathbf{a}\rangle + \beta|\mathbf{b}\rangle$ then $\langle\mathbf{d}|\mathbf{c}\rangle = \alpha\langle\mathbf{d}|\mathbf{a}\rangle + \beta\langle\mathbf{d}|\mathbf{b}\rangle$
- $|\mathbf{a}\rangle$ and $|\mathbf{b}\rangle$ are *orthogonal* if $\langle\mathbf{a}|\mathbf{b}\rangle = \langle\mathbf{b}|\mathbf{a}\rangle = 0$

3D vectors : simple example of a 1D matrix

- Take a (real) 3D coordinate system with orthogonal unit vectors $|\mathbf{e_x}\rangle, |\mathbf{e_y}\rangle, |\mathbf{e_z}\rangle \quad (\equiv \;\; \underline{\mathbf{i}}, \; \underline{\mathbf{j}}, \; \underline{\mathbf{k}})$
- Represent vector $|\mathbf{a}\rangle$ in the form of a *one dimensional column matrix*.

$$|\mathbf{a}\rangle \;\equiv\; \underline{\mathbf{a}} \;=\; \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} \tag{2}$$

  where $|\mathbf{a}\rangle \;=\; a_x|\mathbf{e_x}\rangle + a_y|\mathbf{e_y}\rangle + a_z|\mathbf{e_z}\rangle$.
- Define the *transpose* of the vector (matrix) by interchanging rows and columns:

$$\langle\mathbf{a}| \;\equiv\; \underline{\mathbf{a}}^T \;=\; \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}^T = (a_x, a_y, a_z). \tag{3}$$

- The scalar product of two vectors $\underline{\mathbf{a}}$ and $\underline{\mathbf{b}}$ then becomes:

$$\langle\mathbf{a}|\mathbf{b}\rangle \;\equiv\; \underline{\mathbf{a}}^T . \underline{\mathbf{b}} \;=\; (a_x, a_y, a_z). \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix} \;=\; a_x b_x \;+\; a_y b_y \;+\; a_z b_z. \tag{4}$$

The scalar (inner) product : imaginary vectors

Now consider IMAGINARY vectors

- In complex vector spaces with $|\mathbf{a}\rangle$ complex we want $|\underline{\mathbf{a}}|$ to be a “length” (a real and positive number).
  Hence $\langle\mathbf{a}|\mathbf{a}\rangle$ must become $\underline{\mathbf{a}}^\dagger.\underline{\mathbf{a}}$ where $\underline{\mathbf{a}}^\dagger = \underline{\mathbf{a}}^{T*}$
  (transpose, then complex conjugate) :
  $|\underline{\mathbf{a}}| = \sqrt{\underline{\mathbf{a}}^\dagger.\underline{\mathbf{a}}} = \sqrt{\sum_{i=1}^{N} a_i^* a_i}$
- The inner product of $|\mathbf{a}\rangle$, $|\mathbf{b}\rangle$ is again represented by $\langle\mathbf{a}|\mathbf{b}\rangle$ and is now a complex number.
- If $|\mathbf{a}\rangle$, $|\mathbf{b}\rangle$ are complex, inner product becomes:

$$\langle\mathbf{a}|\mathbf{b}\rangle = \underline{\mathbf{a}}^\dagger.\underline{\mathbf{b}} = (a_1^*, a_2^*, \ldots a_N^*).\begin{pmatrix} b_1 \\ b_2 \\ . \\ b_N \end{pmatrix} = \sum_{i=1}^{N} a_i^* b_i \qquad (5)$$

Other properties:

- $\langle\mathbf{a}|\mathbf{b}\rangle \neq \langle\mathbf{b}|\mathbf{a}\rangle$! For imaginary vectors $\langle\mathbf{a}|\mathbf{b}\rangle = \langle\mathbf{b}|\mathbf{a}\rangle^*$
- Again if $|\mathbf{c}\rangle = \alpha|\mathbf{a}\rangle + \beta|\mathbf{b}\rangle$ then $\langle\mathbf{d}|\mathbf{c}\rangle = \alpha\langle\mathbf{d}|\mathbf{a}\rangle + \beta\langle\mathbf{d}|\mathbf{b}\rangle$
- $|\mathbf{a}\rangle$ and $|\mathbf{b}\rangle$ are *orthogonal* if $\langle\mathbf{a}|\mathbf{b}\rangle = \langle\mathbf{b}|\mathbf{a}\rangle = 0$

7.2 Inner product & basis vectors

- Basis vectors which are mutually orthogonal and each having *norm* $= 1$ are said to be *orthonormal*:
  i.e. $\langle \mathbf{e_i} | \mathbf{e_j} \rangle = \delta_{ij}$ where δ_{ij} is the "Kronecker delta"
  - δ_{ij} = 1 for i = j, δ_{ij} = 0 for i $\neq$ j.
- The inner product of $|\mathbf{a}\rangle$ with the jth element of the orthonormal basis vector $|\mathbf{e_j}\rangle$ will "project out" the jth component of $|\mathbf{a}\rangle$:
  - $|\mathbf{a}\rangle = \sum_{i=1}^{N} a_i |\mathbf{e_i}\rangle$
  - $\langle \mathbf{e_j} | \mathbf{a} \rangle = \sum_{i=1}^{N} \langle \mathbf{e_j} | a_i \mathbf{e_i} \rangle = \sum_{i=1}^{N} a_i \langle \mathbf{e_j} | \mathbf{e_i} \rangle = a_j$
- If the basis vectors defining $|\mathbf{a}\rangle$ and $|\mathbf{b}\rangle$ are orthonormal: then, the inner product is :

$$\begin{aligned}\langle \mathbf{a} | \mathbf{b} \rangle &= \langle a_1 \mathbf{e_1} + \cdots + a_N \mathbf{e_N} | b_1 \mathbf{e_1} + \cdots + b_N \mathbf{e_N} \rangle \\ &= \sum_{i=1}^{N} a_i^* b_i \langle \mathbf{e_i} | \mathbf{e_i} \rangle + \sum_{i=1}^{N} \sum_{j \neq i}^{N} a_i^* b_j \langle \mathbf{e_i} | \mathbf{e_j} \rangle \\ &= \sum_{i=1}^{N} a_i^* b_i\end{aligned}$$

So orthornormality of the basis vectors is important.

7.3 Dual vectors and dual vector spaces

- A "linear map" (or "linear functional") is an operation that takes a real (or complex) vector as input and gives a real $\Re$ (or complex $\Im$) number as output.
- Every vector space V has a "dual" vector space V^* : the space of all possible linear maps from $V \to \Re$ or $V \to \Im$.
- The most common example of a dual vector space is if V is the space of real column vectors, V^* is then the space of real row vectors : inner product maps V to a real number $\Re$

$$\langle \mathbf{a}|\mathbf{b}\rangle = \underline{\mathbf{a}}^T.\underline{\mathbf{b}} = (a_1, a_2, \ldots a_N).\begin{pmatrix} b_1 \\ b_2 \\ . \\ b_N \end{pmatrix} = \sum_{i=1}^{N} a_i b_i \tag{6}$$

$\rightarrow$ $\underline{\mathbf{a}}^T$ is the *dual vector* of $\underline{\mathbf{a}}$

- If $|\mathbf{a}\rangle$, $|\mathbf{b}\rangle$ are complex, inner product maps V to an imaginary number $\Im$

$$\langle \mathbf{a}|\mathbf{b}\rangle = \underline{\mathbf{a}}^\dagger.\underline{\mathbf{b}} = (a_1^*, a_2^*, \ldots a_N^*).\begin{pmatrix} b_1 \\ b_2 \\ . \\ b_N \end{pmatrix} = \sum_{i=1}^{N} a_i^* b_i \tag{7}$$

$\rightarrow$ *Now* $\underline{\mathbf{a}}^\dagger$ is the *dual vector* of $\underline{\mathbf{a}}$

- Note that the dual vector of $\lambda\underline{\mathbf{a}}$ is $\lambda^*\underline{\mathbf{a}}^\dagger$ (λ is a scalar)

7.4 Linear operators

A *linear operator* $\mathbf{A}$ transforms a vector $|\mathbf{x}\rangle$ to give another vector $|\mathbf{y}\rangle$

$$|\mathbf{y}\rangle = \mathbf{A}|\mathbf{x}\rangle$$

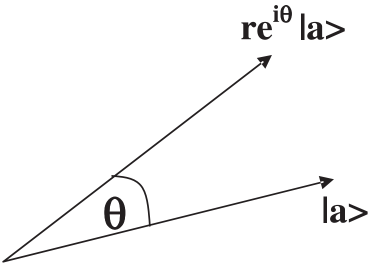
$\mathbf{A}$ is *linear* because of the following requirements:

- For vectors $|\mathbf{a}\rangle$ and $|\mathbf{b}\rangle$ and scalars λ and μ
  $\mathbf{A}(\lambda|\mathbf{a}\rangle + \mu|\mathbf{b}\rangle) = \lambda\mathbf{A}|\mathbf{a}\rangle + \mu\mathbf{A}|\mathbf{b}\rangle$ (associative and distributive).
- For linear operators $\mathbf{A}, \mathbf{B}, \mathbf{C}$ where
  $\mathbf{C} = \mathbf{A} + \mathbf{B}$
  $\Rightarrow \mathbf{C}|\mathbf{a}\rangle = (\mathbf{A} + \mathbf{B})|\mathbf{a}\rangle = (\mathbf{B} + \mathbf{A})|\mathbf{a}\rangle = \mathbf{A}|\mathbf{a}\rangle + \mathbf{B}|\mathbf{a}\rangle$ (distributive)
- For linear operators $\mathbf{D} = \mathbf{A}.\mathbf{B}$
  $\mathbf{D}|\mathbf{a}\rangle = (\mathbf{AB})|\mathbf{a}\rangle = \mathbf{A}(\mathbf{B}|\mathbf{a}\rangle)$
- BUT note that *in general* $(\mathbf{AB})|\mathbf{a}\rangle \neq (\mathbf{BA})|\mathbf{a}\rangle$
  i.e. not commutative.
  This is maybe not so obvious, so more about this when we cover matrices.

7.4.1 Examples of linear operators

- Momentum operator in quantum mechanics $-i\hbar\frac{d}{dx}$
  $-i\hbar\frac{d}{dx}(|\mathbf{a}\rangle + |\mathbf{b}\rangle) = -i\hbar\frac{d}{dx}|\mathbf{a}\rangle \; -i\hbar\frac{d}{dx}|\mathbf{b}\rangle$

- $|\mathbf{b}\rangle = \mathbf{A}|\mathbf{a}\rangle$ where $|\mathbf{a}\rangle$ is a simple complex number and $\mathbf{A} = re^{i\theta}$. The operator transforms $|\mathbf{a}\rangle$ by scaling it by a factor r and rotating it by an angle θ in the Argand diagram.



- Example of an operator that in *not* linear: $A|\mathbf{a}\rangle = |\mathbf{a}\rangle^2$ since
  $A(|\mathbf{a}\rangle + |\mathbf{b}\rangle) = (|\mathbf{a}\rangle + |\mathbf{b}\rangle)^2 \neq A|\mathbf{a}\rangle + A|\mathbf{b}\rangle \;\; [= |\mathbf{a}\rangle^2 + |\mathbf{b}\rangle^2]$

7.5 What is a matrix [1]

- Definition :
  A matrix is a linear operator forming an array of numbers which transforms vectors from an n-dimensional vector space (for which there is a basis $|\mathbf{e_i}\rangle$ with $i = 1, 2, \cdots, n$)) into vectors belonging to an m-dimensional vector space (with a basis $|\mathbf{e_i'}\rangle$ with $i = 1, 2, \cdots, m$)).

What is a matrix [2]

- Alternatively :
  A matrix is made up of a set of numbers (or operators), arranged in rows and columns :-
  - A 2D matrix with m rows and n columns has *size* m $\times$ n.

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{pmatrix} \tag{8}$$

  - The matrix elements of A are A_{ij} (i.e. the ith row and jth column - components of a linear operator). The A_{ij} can also be complex.
- Note the following:
  - A 1 $\times$ 1 matrix is a scalar.
  - An $m \times 1$ (1D) matrix is a column vector; a $1 \times n$ matrix is a row vector.
  - If $m = n$, then A is an $n \times n$ *square* matrix of *order* n.