Classical Mechanics

LECTURE 6: THE CENTRE OF MASS

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MT 2016

OUTLINE : 6. THE CENTRE OF MASS

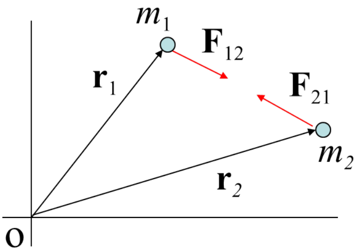
6.1 Lab & CM frames of reference

From hereon we will deal with 2 inertial frames:

- The Laboratory frame: this is the frame where measurements are actually made
- The centre of mass frame: this is the frame where the centre of mass of the system is at rest and where the total momentum of the system is zero

6.2 Internal forces and reduced mass

- Internal forces only:
  $\underline{\mathbf{F}}_{12} = m_1 \ddot{\underline{\mathbf{r}}}_1 \;\; ; \underline{\mathbf{F}}_{21} = m_2 \ddot{\underline{\mathbf{r}}}_2$
  Then $\ddot{\underline{\mathbf{r}}}_2 - \ddot{\underline{\mathbf{r}}}_1 = \frac{\underline{\mathbf{F}}_{21}}{m_2} - \frac{\underline{\mathbf{F}}_{12}}{m_1}$
  NIII : $\underline{\mathbf{F}}_{21} = -\underline{\mathbf{F}}_{12} = \underline{\mathbf{F}}_{int}$



- Define $\underline{\mathbf{r}} = \underline{\mathbf{r}}_2 - \underline{\mathbf{r}}_1 \rightarrow \dot{\underline{\mathbf{r}}} = \dot{\underline{\mathbf{r}}}_2 - \dot{\underline{\mathbf{r}}}_1$
- $\underline{\mathbf{F}}_{int} \left(\frac{1}{m_1} + \frac{1}{m_2} \right) = \ddot{\underline{\mathbf{r}}}$
- Define $\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2} \quad \rightarrow \quad \underline{\mathbf{F}}_{int} = \mu \ddot{\underline{\mathbf{r}}}$
  $\mu = \frac{m_1 m_2}{m_1 + m_2}$ is the *reduced mass* of the system

This defines the equation of motion of the relative motion of the particles under internal forces, with position vector $\underline{\mathbf{r}}$ & mass μ

6.3 The Centre of Mass

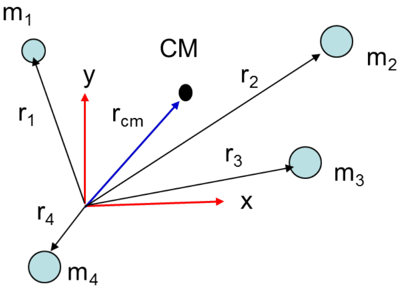
The centre of mass (CM) is the point where the mass-weighted position vectors (moments) relative to the point sum to zero ; the CM is the mean location of a distribution of mass in space.

- Take a system of n particles, each with mass m_i located at positions $\underline{\mathbf{r_i}}$, the position vector of the CM is defined by:

  $\sum_{i=1}^{n} m_i(\underline{\mathbf{r_i}} - \underline{\mathbf{r_{cm}}}) = 0$
- Solve for $\underline{\mathbf{r_{cm}}}$:

  $\underline{\mathbf{r_{cm}}} = \frac{1}{M} \sum_{i=1}^{n} m_i \, \underline{\mathbf{r_i}}$

  where $M = \sum_{i=1}^{n} m_i$

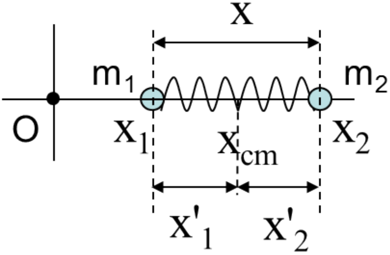


Example : SHM of two connected masses in 1D

SHM between two masses m_1 and m_2 connected by a spring

- $x = x_2 - x_1$; Natural length L
- $F_{int} = -k(x - L) = \mu \ddot{x}$

  $(\mu = \frac{m_1 m_2}{m_1 + m_2} =$ reduced mass$)$
- $\ddot{x} + \frac{k}{\mu}(x - L) = 0$



Solution: $x = x_0 \cos(\omega t + \phi) + L$

where $\omega = \sqrt{\frac{k}{\mu}}$

With respect to the CM:

- $x_{CM} = \frac{m_1 x_1 + m_2 x_2}{M}$ where $M = m_1 + m_2$
- $x_1' = x_1 - x_{cm} = \frac{M x_1 - m_1 x_1 - m_2 x_2}{M} = -\frac{m_2 x}{M}$
- $x_2' = x_2 - x_{cm} = \frac{M x_2 - m_1 x_1 - m_2 x_2}{M} = \frac{m_1 x}{M}$

Eg. take $m_1 = m_2 = m \rightarrow \omega = \sqrt{\frac{2k}{m}}$; $x_1' = -\frac{1}{2}x$, $x_2' = \frac{1}{2}x$

6.3.1 CM of a continuous volume

If the mass distribution is continuous with density $\rho(\underline{\mathbf{r}})$ inside a volume V, then:

- $\sum_{i=1}^{n} m_i(\underline{\mathbf{r}}_\mathbf{i} - \underline{\mathbf{r}}_\mathbf{cm}) = 0$

  becomes

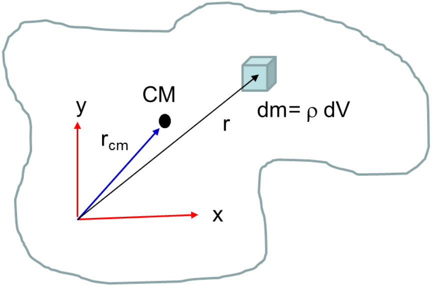
  $\int_V \rho(\underline{\mathbf{r}})(\underline{\mathbf{r}} - \underline{\mathbf{r}}_\mathbf{cm})dV = 0$

  where $dm = \rho(\underline{\mathbf{r}})dV$

- Solve for r_{cm}

  $\underline{\mathbf{r}}_\mathbf{cm} = \frac{1}{M}\int_V \rho(\underline{\mathbf{r}})\ \underline{\mathbf{r}}\ dV$

  where M is the total mass in the volume



Example: the CM of Mount Ranier

Mount Ranier has approximately the shape of a cone (assume uniform density) and its height is 4400 m. At what height is the centre of mass?

We have cylindrical symmetry - just need to consider the y direction. Integrate from top ($y = 0$) to bottom ($y = h$)

- $y_{cm} = \frac{\int_0^h y dm}{M} = \frac{\int_0^h y dm}{\int_0^h dm}$

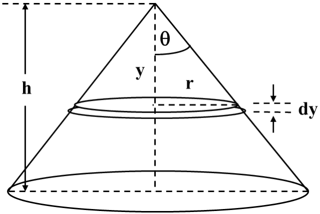
  $dm = \rho(\pi r^2) dy = \rho(\pi y^2 \tan^2 \theta) dy$

- $y_{cm} = \frac{\int_0^h y \rho(\pi y^2 \tan^2 \theta) dy}{\int_0^h \rho(\pi y^2 \tan^2 \theta) dy}$

  $y_{cm} = \frac{\int_0^h y^3 dy}{\int_0^h y^2 dy} = \frac{3h^4}{4h^3} = \frac{3h}{4}$

  (measured from the top)

- $h = 4400m \rightarrow \; y_{cm} = 1100m$ above the base



6.3.2 Velocity in the Centre of Mass frame

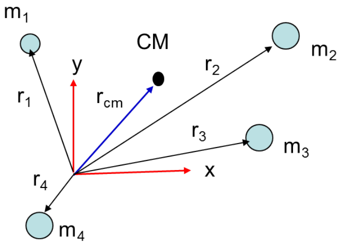
- The position of the centre of mass is given by:

  $\underline{\mathbf{r_{cm}}} = \frac{1}{M}\sum_{i=1}^{n} m_i\, \underline{\mathbf{r_i}}$

  where $M = \sum_{i=1}^{n} m_i$

- The velocity of the CM:

  $\underline{\mathbf{v_{cm}}} = \underline{\dot{\mathbf{r}}_{\mathbf{cm}}} = \frac{1}{M}\sum_{i=1}^{n} m_i\, \underline{\dot{\mathbf{r}}_{\mathbf{i}}}$



  - In the Lab frame, total momentum $\underline{\mathbf{p}}_{tot}$:

    $\underline{\mathbf{p}}_{tot} = \sum_{i=1}^{n} m_i\, \underline{\dot{\mathbf{r}}_{\mathbf{i}}} = M\underline{\mathbf{v}}_{\mathbf{cm}}$

Hence the total momentum of a system in the Lab frame is equivalent to that of a single particle having a mass M and moving at a velocity $\underline{\mathbf{v}}_{\mathbf{cm}}$

6.3.3 Momentum in the CM frame

- Velocity of the CM: $\underline{\mathbf{v}}_{\mathbf{cm}} = \underline{\dot{\mathbf{r}}}_{\mathbf{cm}} = \frac{\sum_{i=1}^{n} m_i \; \underline{\dot{\mathbf{r}}}_{\mathbf{i}}}{\sum_i m_i} = \frac{\sum_{i=1}^{n} m_i \; \underline{\mathbf{v}}_{\mathbf{i}}}{\sum_i m_i}$
- Velocity of a body in the CM relative to Lab $\underline{\mathbf{v}}'_{\mathbf{i}} = \underline{\mathbf{v}}_{\mathbf{i}} - \underline{\mathbf{v}}_{\mathbf{cm}}$
- The total momentum of the system of particles in the CM:
- $\sum_i \underline{\mathbf{p}}'_{\mathbf{i}} = \sum_i m_i \underline{\mathbf{v}}'_{\mathbf{i}} = \sum_i m_i (\underline{\mathbf{v}}_{\mathbf{i}} - \underline{\mathbf{v}}_{\mathbf{cm}})$
  $= \sum_i m_i \underline{\mathbf{v}}_{\mathbf{i}} - \sum_i m_i \frac{\sum_j m_j \underline{\mathbf{v}}_{\mathbf{j}}}{\sum_j m_j} = \sum_i m_i \underline{\mathbf{v}}_{\mathbf{i}} - \sum_j m_j \underline{\mathbf{v}}_{\mathbf{j}} = 0$

Hence the total momentum of a system of particles in the CM frame is equal to zero

- In addition, the total energy of the system is a minimum compared to all other inertial reference frames.

6.3.4 Motion of CM under external forces

- Force on particle i: $m_i \ddot{\underline{\mathbf{r}}}_i = \underline{\mathbf{F}}_\mathbf{i}{}^{ext} + \underline{\mathbf{F}}_\mathbf{i}{}^{int}$

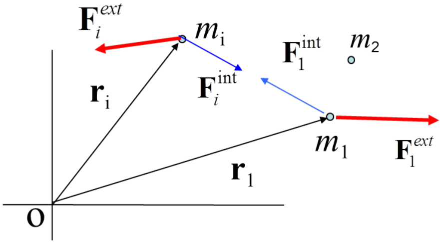
- $$\underbrace{\sum_i^n m_i \ddot{\underline{\mathbf{r}}}_i}_{\text{all masses}} = \underbrace{\sum_i^n \underline{\mathbf{F}}_\mathbf{i}{}^{ext}}_{\text{external forces}} + \underbrace{\sum_i^n \underline{\mathbf{F}}_\mathbf{i}{}^{int}}_{\text{internal forces = zero}} = \sum_i^n \underline{\mathbf{F}}_\mathbf{i}{}^{ext}$$

- $\underline{\mathbf{r}}_{CM} = \sum_i \frac{m_i \underline{\mathbf{r}}_\mathbf{i}}{M}$

  where $M = \sum_i m_i$

- $\ddot{\underline{\mathbf{r}}}_{CM} = \sum_i \frac{m_i \ddot{\underline{\mathbf{r}}}_i}{M}$

  $\rightarrow M \ddot{\underline{\mathbf{r}}}_{CM} = \sum_i \underline{\mathbf{F}}_\mathbf{i}{}^{ext}$



The motion of the system is equivalent to that of a single particle having a mass *M* acted on by the sum of external forces

(The CM moves at constant velocity if no external forces)

6.4 Kinetic energy and the CM

- Lab kinetic energy : $T_{Lab} = \frac{1}{2}\sum m_i \underline{\mathbf{v}}_i^2 \quad ; \quad \underline{\mathbf{v}}'_i = \underline{\mathbf{v}}_i - \underline{\mathbf{v}}_{CM}$

  where $\underline{\mathbf{v}}'_i$ is velocity of particle i in the CM

- $T_{Lab} = \frac{1}{2}\sum m_i \underline{\mathbf{v}}_i'^2 + \sum m_i \underline{\mathbf{v}}'_i \cdot \underline{\mathbf{v}}_{CM} + \frac{1}{2}\sum m_i \underline{\mathbf{v}}_{CM}^2$

- But $\sum m_i \underline{\mathbf{v}}'_i \cdot \underline{\mathbf{v}}_{CM} = \underbrace{\frac{\sum m_i \underline{\mathbf{v}}'_i}{M}}_{=\,\underline{\mathbf{0}}} \cdot M\,\underline{\mathbf{v}}_{CM} = 0$

$\rightarrow \quad T_{Lab} = T_{CM} + \frac{1}{2}Mv_{cm}^2$

The kinetic energy in the Lab frame is equal to the kinetic energy in CM + the kinetic energy of the CM