LECTURE 6: LINEAR VECTOR SPACES, BASIS VECTORS AND LINEAR INDEPENDENCE

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Outline: 6. LINEAR VECTOR SPACES, BASIS VECTORS AND LINEAR INDEPENDENCE

6.1 Linear vector spaces

- Up to now we have considered 3D space where the components of a vector are $\underline{\mathbf{a}} = (a_x\underline{\mathbf{i}} + a_y\underline{\mathbf{j}} + a_z\underline{\mathbf{k}})$ with respect to the Cartesian coordinate system.
- We now extend to more abstract spaces which can have an arbitrary number of N dimensions. A vector is still an “arrow” in this *linear vector space*.
- An example of a linear vector space could be any system where N variables can uniquely specify the state - e.g. $N = 5$: [energy, angle, mass, charge, spin] of a moving body.

6.1.1 Notation

- To represent an object in linear vector space we now introduce *Dirac notation* $\rightarrow \quad |\mathbf{a}\rangle$ (this can of course also represent a “standard” vector in 3D space).
- This notation is widely used in quantum mechanics:
  - “ket” vector $|\mathbf{a}\rangle$ (the standard representation for an object in linear vector space)
  - “bra” vector $\langle\mathbf{a}|$ (the complex conjugate of $|\mathbf{a}\rangle$ and its transpose (see later))
  - “bra-ket” $\langle\mathbf{a}|\mathbf{b}\rangle$ (specifies the “inner product”).

6.1.2 Definitions

Start with a set V of objects (vectors).

Definition – V forms a linear vector space if:

- V is "closed" under addition:
  i.e. if the vectors $|\mathbf{a}\rangle$ and $|\mathbf{b}\rangle$ are elements of V, then the vector $|\mathbf{a}\rangle + |\mathbf{b}\rangle$ is also an element of V.
  (i.e. $|\mathbf{a}\rangle, |\mathbf{b}\rangle \,\epsilon\, V \;\Rightarrow |\mathbf{a}\rangle + |\mathbf{b}\rangle \,\epsilon\, V$).
  Addition must be commutative and associative
  $|\mathbf{a}\rangle + |\mathbf{b}\rangle = |\mathbf{b}\rangle + |\mathbf{a}\rangle$ and
  $(|\mathbf{a}\rangle + |\mathbf{b}\rangle) + |\mathbf{c}\rangle = |\mathbf{a}\rangle + (|\mathbf{b}\rangle + |\mathbf{c}\rangle)$
- V is closed under multiplication by a scalar:
  i.e. if the vector $|\mathbf{a}\rangle$ is an element of V, then the vector $\lambda|\mathbf{a}\rangle$ is also an element of V
  (i.e. $|\mathbf{a}\rangle \,\epsilon\, V \;\Rightarrow \lambda|\mathbf{a}\rangle \,\epsilon\, V$).

Definitions, continued

- Multiplication by a scalar must be associative and distributive.
  - $\lambda(|\mathbf{a}\rangle + |\mathbf{b}\rangle) = \lambda|\mathbf{a}\rangle + \lambda|\mathbf{b}\rangle$ and
  - $(\lambda + \mu)|\mathbf{a}\rangle = \lambda|\mathbf{a}\rangle + \mu|\mathbf{a}\rangle$ and
  - $\lambda(\mu|\mathbf{a}\rangle) = (\lambda\mu)|\mathbf{a}\rangle$
- There exists a *null* (*zero*) *element* $|\mathbf{0}\rangle \in V$
  - $|\mathbf{0}\rangle + |\mathbf{a}\rangle = |\mathbf{a}\rangle$
- Multiplication by unity leaves any vector unchanged
  - $1 \times |\mathbf{a}\rangle = |\mathbf{a}\rangle$
- If the vector $|\mathbf{a}\rangle$ is an element of V, then the vector $|\mathbf{a}'\rangle$ is also an element of V with the property:
  - $|\mathbf{a}\rangle + |\mathbf{a}'\rangle = 0$
  - i.e. $|\mathbf{a}'\rangle = |\mathbf{a}\rangle \times (-1)$

6.1.3 Examples of linear vector spaces

- Obviously spatial 3D vectors form a linear vector space.
- $\Re$ (the real numbers) form a (real) linear vector space. The "vectors" in this space are simply the real numbers. Addition and multiplication (by real numbers) fulfill all the above requirements on a linear vector space. The null element is the number zero.
- $\Im$ (the complex numbers) form a complex linear vector space. The vectors in this space are simply the complex numbers.
- The spherical harmonics which are angular functions $Y_l^m(\theta, \phi)$: a few examples:
  - $Y_0^0(\theta, \phi) = \surd(1/4\pi)$
  - $Y_1^{-1}(\theta, \phi) = \surd(3/8\pi)\sin\theta\ e^{-i\phi}$
  - $Y_1^0(\theta, \phi) = \surd(3/4\pi)\cos\theta$
  - $Y_1^1(\theta, \phi) = \surd(3/8\pi)\sin\theta\ e^{+i\phi}$
  - $Y_2^0(\theta, \phi) = \surd(5/16\pi)(3\cos^2\theta - 1)$ etc.

  (these functions form a "basis" set of wavefunctions in quantum mechanics).

6.2 Linear dependent and independent vectors

- $|\mathbf{a_i}\rangle_{i=1\dots N}$ is a set of N vectors in a given space and α_i is a set of scalars. The linear combination of the vectors $|\mathbf{a_i}\rangle$ with coefficients α_i is given by

$$\alpha_1|\mathbf{a_1}\rangle + \alpha_2|\mathbf{a_2}\rangle + \cdots + \alpha_n|\mathbf{a_n}\rangle = \sum_{i=1}^{N} \alpha_i|\mathbf{a_i}\rangle$$

- Example in a 3D coordinate system: here N = 3 and the vectors $|\mathbf{a_i}\rangle$ can be the three unit (base) vectors (1, 0, 0), (0, 1, 0), (0, 0, 1) and the coefficients are the components (a_x, a_y, a_z).
- A set of vectors is said to be *linearly dependent* if for some α_i

$$\sum_{i=1}^{N} \alpha_i|\mathbf{a_i}\rangle = 0$$

  i.e. this means that at least one vector is redundant, and can be expressed as a linear sum of the others.
- However if $\sum_{i=1}^{N} \alpha_i|\mathbf{a_i}\rangle \neq 0$

  for *any* set of coefficients α_i (other than the trivial case $\alpha_i = 0$, for all i) then the vectors are said to be *linearly independent*.

  i.e. this means that when vectors are linearly independent, none of the vectors can be obtained as a combination of the others.

6.2.1 Examples

a) $|\mathbf{a_1}\rangle$ = (0,1) and $|\mathbf{a_2}\rangle$ = (1,0) are linearly independent (since $\alpha_1|\mathbf{a_1}\rangle + \alpha_2|\mathbf{a_2}\rangle$ can never be 0).

b) $|\mathbf{a_1}\rangle$ = (0,1) and $|\mathbf{a_2}\rangle$ = (1,1) are linearly independent (even though these vectors are not orthogonal).

c) $|\mathbf{a_1}\rangle$ = (0,1) and $|\mathbf{a_2}\rangle$ = (1,0) and $|\mathbf{a_3}\rangle$ = (5,1) are linearly dependent since
$|\mathbf{a_1}\rangle$ + 5$|\mathbf{a_2}\rangle$ - $|\mathbf{a_3}\rangle$ = 0.

d) In a 3D space (1, 0, 0), (0, 1, 0), (0, 0, 1) define a 3D vector space; linear independence is naturally related to the orthogonality of these *base* vectors.

Quantum Mechanics makes fundamental use out of these apparently simple concepts.

6.3 Basis vectors in N dimensions

- If there exists N linearly independent vectors, the vector space is said to be N-dimensional.
- Consider a linear vector space V with a *basis set* of N linearly independent vectors, $|\mathbf{e_1}\rangle, |\mathbf{e_2}\rangle \cdots |\mathbf{e_N}\rangle$.
  Any general vector $|\mathbf{a}\rangle$ can be expressed as a linear sum of base vectors $|\mathbf{e_i}\rangle$ with scalar coefficients a_i :

$$|\mathbf{a}\rangle = a_1|\mathbf{e_1}\rangle + a_2|\mathbf{e_2}\rangle + \cdots + a_N|\mathbf{e_N}\rangle = \sum_{i=1}^{N} a_i|\mathbf{e_i}\rangle$$

- The base vectors are said to form a *complete set* for that space (i.e., any vector of the space can be uniquely expressed as a linear combination of these vectors).
- Base vectors for an N-dimensional space are not unique - *any* set of N linear independent vectors can form a basis for the space.