Classical Mechanics

LECTURE 5:

KINETIC & POTENTIAL ENERGY

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OUTLINE : 5. KINETIC & POTENTIAL ENERGY

5.1 Conservative forces

$$W_{ab} = \int_a^b \underline{\mathbf{F}} \, . \, d\underline{\mathbf{r}} = U(a) - U(b)$$

For a conservative field of force, the work done depends only on the initial and final positions of the particle independent of the path.

The conditions for a conservative force (*all equivalent*) are:

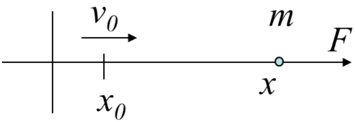
- The force is derived from a (scalar) potential function: $\underline{\mathbf{F}}(\underline{\mathbf{r}}) = -\nabla U \quad \rightarrow \quad F(x) = -\frac{dU}{dx}$ etc.
- There is zero net work by the force when moving a particle around any closed path: $W = \oint_c \underline{\mathbf{F}} \, . \, d\underline{\mathbf{r}} = 0$
- In equivalent vector notation $\underline{\nabla} \times \underline{\mathbf{F}} = 0$

For *any* force: $W_{ab} = \frac{1}{2}mv_b^2 - \frac{1}{2}mv_a^2$

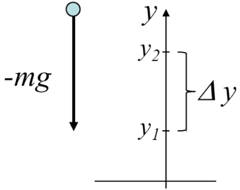
Only for a *conservative* force: $W_{ab} = U(a) - U(b)$

Conservative force: example 1. Constant acceleration

Consider a particle moving under constant force (in 1-D).



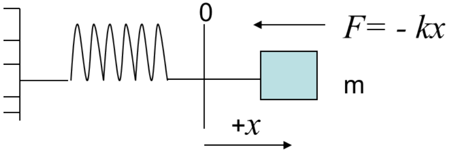
- $F = ma$. Say at $t = 0 \rightarrow x = x_0$ and $v = v_0$
- $T_2 + U_2 = T_1 + U_1$ (the total energy is conserved)
- $\frac{1}{2}mv^2 - (ma)x = \frac{1}{2}mv_0^2 - (ma)x_0 =$ constant
- $v^2 = v_0^2 + 2a(x - x_0)$



- Gravitational potential energy
- $U(\Delta y) = -\int_{y_1}^{y_2} F(y)dy$
- $U(\Delta y) = -\int_{y_1}^{y_2}(-mg)dy = mg(y_2 - y_1)$

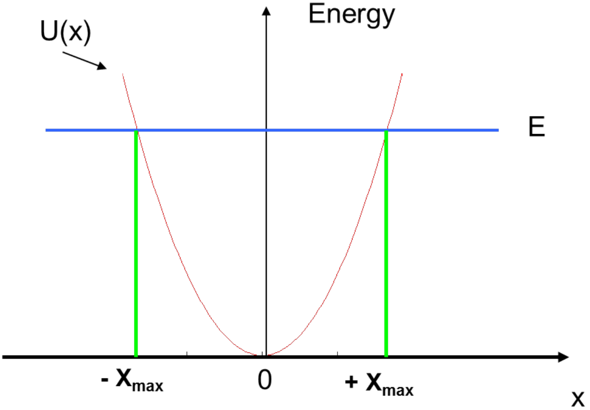
Example 2. Simple harmonic oscillator

Equation of motion: $F = m\frac{d^2x}{dt^2} = -kx$



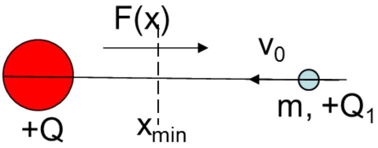
- Potential energy: $U(x) = -\int_0^x F dx = -\int_0^x (-kx) dx = \frac{kx^2}{2}$
- Total energy: $E = T(x) + U(x) = \frac{1}{2}m\dot{x}^2 + \frac{kx^2}{2}$
- Check conservation of energy:
  EOM : $m\ddot{x} + kx = 0 \;\rightarrow$ [multiply by $\dot{x}$] $m\ddot{x}\dot{x} + kx\dot{x} = 0$
  $\rightarrow \; \frac{1}{2}m\frac{d}{dt}(\dot{x}^2) + \frac{1}{2}k\frac{d}{dt}(x^2) = 0$
- Integrate wrt t: $\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = \text{constant} \quad \rightarrow$ i.e. energy conserved.

SHM potential energy curve



- $E = U(x) + \frac{1}{2}mv^2$
- The particle can only reach locations x that satisfy $U < E$

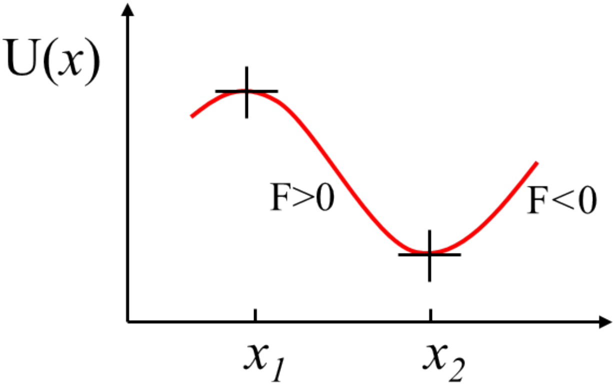
Example 3. Minimum approach of a charge



A particle of mass m and charge $+Q_1$ starts from $x = +\infty$ with velocity v_0. It approaches a fixed charge $+Q$. Calculate its minimum distance of approach x_{min}.

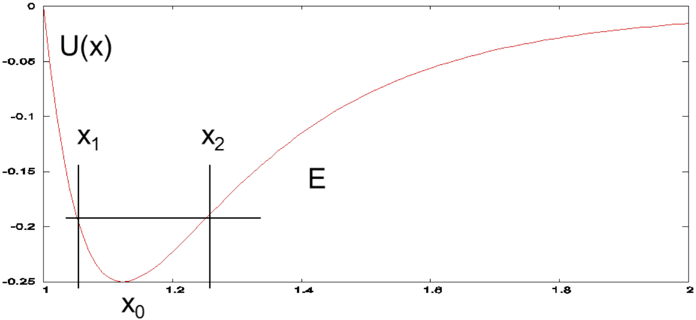
- Force on charge $+Q_1$: $F(x) = +\frac{QQ_1}{4\pi\epsilon_0 x^2}$ (+ve direction)
- Potential energy at point x: $U(x) = -\int_\infty^x F(x)dx = +\frac{QQ_1}{4\pi\epsilon_0 x}$ (where PE = 0 at $x = \infty$)
- Conservation of energy : $\frac{1}{2}mv_0^2 + 0 = \frac{1}{2}mv^2 + U(x)$
- Min. dist. when $v = 0$: $\frac{1}{2}mv_0^2 = \frac{QQ_1}{4\pi\epsilon_0 x_{min}} \rightarrow x_{min} = \frac{QQ_1}{2\pi m\epsilon_0 v_0^2}$

5.2 Potential with turning points



- U is a maximum: *unstable* equilibrium
- U is a minimum: *stable* equilibrium

5.2.1 Oscillation about stable equilibrium

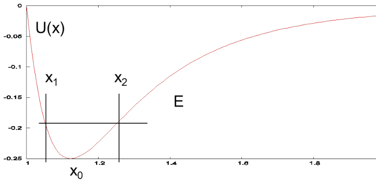


- For SHM : $U(x) = \frac{1}{2}k(x - x_0)^2$
- Taylor expansion about x_0:

$$U(x) = U(x_0) + \underbrace{\left[\frac{dU}{dx}\right]_{x=x_0}}_{=0}(x - x_0) + \frac{1}{2!}\underbrace{\left[\frac{d^2U}{dx^2}\right]_{x=x_0}}_{=k}(x - x_0)^2 + \ldots$$

Example: The Lennard-Jones potential

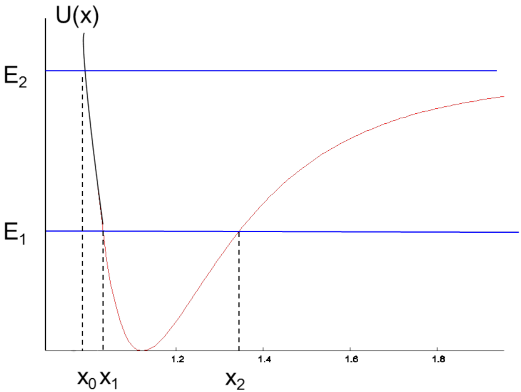
The Lennard-Jones potential describes the potential energy between two atoms in a molecule: $U(x) = \epsilon[(x_0/x)^{12} - 2(x_0/x)^6]$ (ϵ and x_0 are constants and x is the distance between the atoms).



Show that the motion for small displacements about the minimum is simple harmonic and find its frequency.

- $U(x) = U(x_0) + \left[\frac{dU}{dx}\right]_{x=x_0}(x - x_0) + \frac{1}{2!}\left[\frac{d^2U}{dx^2}\right]_{x=x_0}(x - x_0)^2 + \dots$
- $U(x_0) = \epsilon[(x_0/x)^{12} - 2(x_0/x)^6]_{x=x_0} = -\epsilon$
- $\frac{dU(x)}{dx}|_{x=x_0} = 12\epsilon[-\frac{1}{x_0}(x_0/x)^{13} + \frac{1}{x_0}(x_0/x)^7]_{x=x_0} = 0$ as expected.
- $\frac{d^2U(x)}{dx^2}|_{x=x_0} = \frac{12\epsilon}{x_0^2}[13(x_0/x)^{14} - 7(x_0/x)^8]_{x=x_0} = \frac{72\epsilon}{x_0^2}$
- Hence $U(x) \approx -\epsilon + \frac{72\epsilon}{2!x_0^2}(x - x_0)^2$
- $F(x) = -\frac{dU}{dx} \approx -\frac{1}{2} \times 2(\frac{72\epsilon}{x_0^2})(x - x_0) = -k(x - x_0)$ SHM about x_0
- Angular frequency of small oscillations : $\omega^2 = \frac{k}{m} = \frac{72\epsilon}{mx_0^2}$

5.2.2 Bounded and unbounded potentials



- Bounded motion : $E = E_1$: x constrained $x_1 < x < x_2$
- Unbounded motion : $E = E_2$: x unconstrained at high x $x_0 < x < \infty$