Classical Mechanics

LECTURE 4: NEWTON'S LAWS

Prof. N. Harnew
University of Oxford

MT 2016

OUTLINE : 4. NEWTON'S LAWS

4.1 Newton's Second Law

4.2 Newton's Third Law

4.3 Energy conservation in one dimension

4.1 Newton's Second Law

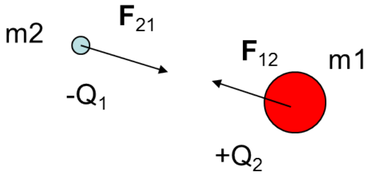
The rate of change of momentum of a body is equal to the applied force on the body.

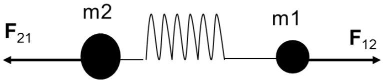
- $\underline{\mathbf{F}} = \frac{d\underline{\mathbf{p}}}{dt} = m\underline{\mathbf{a}}$ where $\underline{\mathbf{p}} = m\underline{\mathbf{v}}$
- In components: $(F_x, F_y, F_z) = m(a_x, a_y, a_z)$
- Assuming constant mass, we can define *the equation of motion* in 1D : $F = \frac{d(mv)}{dt} = m\frac{d^2x}{dt^2}$
- We require two initial conditions *for a unique solution*: e.g. $v = v_0$ at $t = 0$ and $x = x_0$ at $t = 0$

We shall later solve the EOM for three examples:
(i) F = constant, (ii) $F \propto -v$, (iii) $F \propto -x$

4.2 Newton's Third Law

Action and reaction forces are equal in magnitude and opposite in direction.



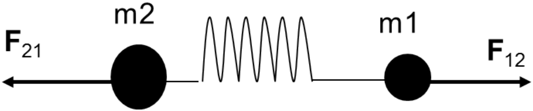


Compressed spring

Electrostatic interaction

$$\underline{\mathbf{F}}_{12} = -\underline{\mathbf{F}}_{21}$$

Conservation of momentum



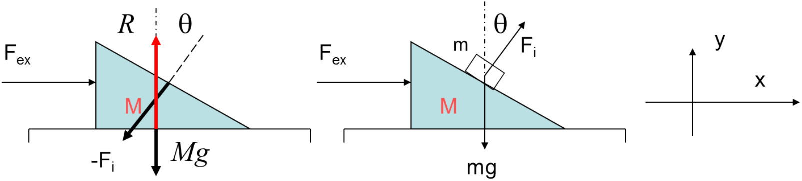
Compressed spring

- $\underline{\mathbf{F}}_{12} = m_1 \underline{\mathbf{a}}_1 = \frac{d\underline{\mathbf{P}}_1}{dt}$ and $\underline{\mathbf{F}}_{21} = m_2 \underline{\mathbf{a}}_2 = \frac{d\underline{\mathbf{P}}_2}{dt}$
- $\underline{\mathbf{F}}_{12} + \underline{\mathbf{F}}_{21} = \frac{d}{dt}(\underline{\mathbf{P}}_1 + \underline{\mathbf{P}}_2) = 0$ (Newton III)
- Therefore $(\underline{\mathbf{P}}_1 + \underline{\mathbf{P}}_2)$ = constant

In an isolated system, the total momentum is conserved.

Newton II : Example 1. E.O.M. under constant force

How fast should we accelerate the triangular wedge to keep the block *m* stationary on the wedge?



Forces on wedge:

- Horizontal: $F_{ex} - F_i \sin\theta = MA_x$
- Vertical: $R - F_i \cos\theta - Mg = 0$

Forces on block:

- Horizontal: $F_i \sin\theta = ma_x$
- Vertical: $F_i \cos\theta - mg = ma_y$

For block to remain at the same place $A_x = a_x$ and $a_y = 0$

- $F_i = \frac{mg}{\cos\theta}$ and $a_x = g\tan\theta = A_x$
- Hence $F_{ex} = Mg\tan\theta + mg\tan\theta = (m+M)g\tan\theta$

Example constant force, continued

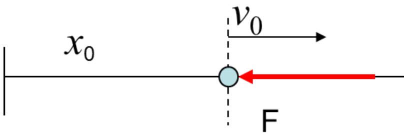
What is the internal force that the blocks apply on each other and the reaction force by the ground on M?

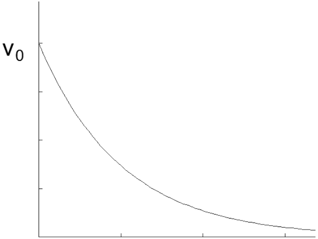
From before:

- $F_i = \frac{mg}{\cos\theta}$
- $R - F_i \cos\theta - Mg = 0$
- Hence: $R = F_i \cos\theta + Mg = (m + M)g$

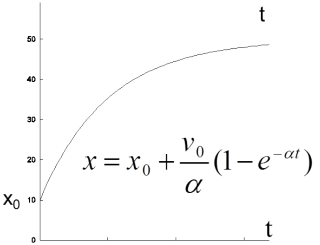
Example 2. Force proportional to velocity

Solve the equation of motion for the case $F = -\beta v \ \ (\beta > 0)$ with $x = x_0$ and $v = v_0$ at $t = 0$



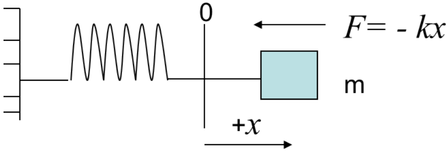


- $m\frac{dv}{dt} = -\beta v$
- $\frac{dv}{dt} = -\alpha v$ where $\alpha = \frac{\beta}{m}$
- $\int_{v_0}^{v} \frac{dv}{v} = -\alpha \int_0^t dt \quad \rightarrow \quad v = v_0 e^{-\alpha t}$
- $v = \frac{dx}{dt} \rightarrow \int_{x_0}^{x} dx = \int_0^t v dt = \int_0^t v_0 e^{-\alpha t} dt$
- $x - x_0 = -\frac{v_0}{\alpha} e^{-\alpha t} + \frac{v_0}{\alpha}$
- $x = x_0 + \frac{v_0}{\alpha}(1 - e^{-\alpha t})$
- When $t \rightarrow \infty, \quad x \rightarrow x_0 + \frac{v_0}{\alpha}$

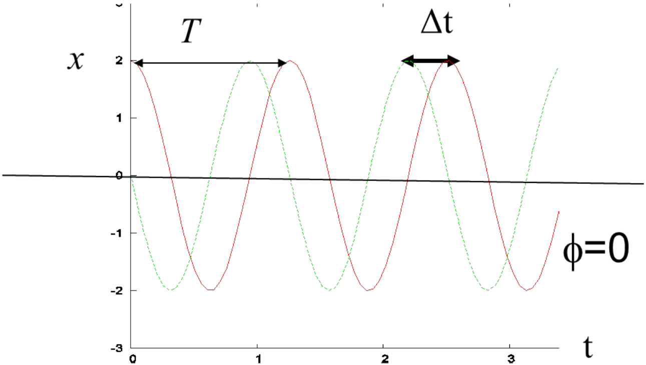


Example 3. Force proportional to position: simple harmonic oscillator

Solving the equation of motion for the case $F = m\frac{d^2x}{dt^2} = -kx$



- $m\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$; trial solution $x = A\cos\omega t + B\sin\omega t$
  $\rightarrow \dot{x} = -A\omega\sin\omega t + B\omega\cos\omega t$; $\ddot{x} = -A\omega^2\cos\omega t - B\omega^2\sin\omega t$
- $\ddot{x} = -\omega^2 x \rightarrow \omega^2 = \frac{k}{m}$
- Alternatively $x = x_0\cos(\omega t + \phi)$ (or $x = x_0 Re[e^{i(\alpha t+\phi)}]$)
- Expand : $x = x_0(\cos(\omega t)\cos\phi - \sin(\omega t)\sin\phi)$
  $A = x_0\cos\phi$; $B = -x_0\sin\phi \rightarrow x_0^2 = A^2 + B^2$; $\tan\phi = -B/A$
- x_0 = amplitude, ϕ = phase, ω = angular frequency ($T = \frac{2\pi}{\omega}$)



- $x = x_0 \cos(\omega t + \phi)$
- $\omega = \sqrt{\frac{k}{m}}$
- $\phi = \omega \Delta t$

4.3 Energy conservation in one dimension

Work done on a body by a force F

- $W = \int_{x_1}^{x_2} F(x)dx = m\int_{x_1}^{x_2} \frac{dv}{dt}dx$
- We can write: $\frac{dv}{dt}dx = \frac{dx}{dt}dv = vdv$
  hence $\int_{x_1}^{x_2} F(x)dx = m\int_{v_1}^{v_2} vdv = \frac{1}{2}m(v_2^2 - v_1^2) = T_2 - T_1$
- Now introduce an arbitrary reference point x_0
  $\int_{x_1}^{x_2} Fdx = \int_{x_0}^{x_2} Fdx - \int_{x_0}^{x_1} Fdx$ defines a *conservative* force
  hence $T_2 + [-\int_{x_0}^{x_2} Fdx] = T_1 + [-\int_{x_0}^{x_1} Fdx]$
- We define the *potential energy* $U(x)$ at a point x :
  $U(x) - U(x_0) = -\int_{x_0}^{x} Fdx$ and hence
  $T_2 + U_2 = T_1 + U_1$ (total energy PE + KE conserved)
- Note the minus sign. The potential energy (relative to a reference point) is always the *negative* of the work done by the force $\rightarrow$ $F(x) = -\frac{dU}{dx}$