LECTURE 4:

VECTOR GEOMETRY : REPRESENTATION OF LINES

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MT 2012

Outline: 4. VECTOR GEOMETRY

4.1 Representation of lines in vector form

4.1.1 From vector to Cartesian form

4.1.2 From Cartesian to vector equation

4.2 Distance from a point to a line

4.2.1 Example

4.3 Minimum distance from a line to a line

4.3.1 Example

4.3.2 Angle between the crossing lines

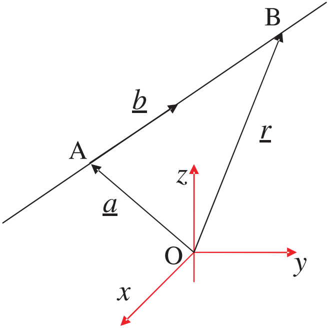
4.3.3 Finding the coordinate at which lines cross

4.1 Representation of lines in vector form

- A very useful application of vectors - $lines$ can be represented in vector form.
- Point A is any fixed position on the line with position vector $\underline{\mathbf{a}}$. Line direction is defined by vector $\underline{\mathbf{b}}$. Position vector $\underline{\mathbf{r}}$ is a general point on the line.
- The equation of the line is then written by:

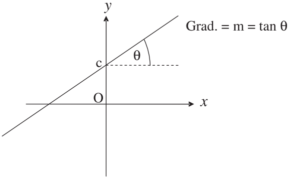
$$\underline{\mathbf{r}} = \underline{\mathbf{a}} + \lambda \underline{\mathbf{b}}$$

where λ takes all values to give all positions on the line.



4.1.1 From vector to Cartesian form

- Equation of line, $\underline{\mathbf{r}} - \underline{\mathbf{a}} = \lambda \underline{\mathbf{b}}$
- Taking components:
  $\frac{x-a_x}{b_x} = \frac{y-a_y}{b_y} = \frac{z-a_z}{b_x} = \lambda$
- From first 2 equations:
  $y = x(\frac{b_y}{b_x}) + (a_y - \frac{a_x b_y}{b_x})$
  $\equiv y = mx + c$ gets the Cartesian representation.



Alternative vector equation

- $\underline{\mathbf{r}} - \underline{\mathbf{a}} = \lambda \underline{\mathbf{b}}$
  cross-product both sides by $\underline{\mathbf{b}}$ $\rightarrow$
  $(\underline{\mathbf{r}} - \underline{\mathbf{a}}) \times \underline{\mathbf{b}} = \lambda(\underline{\mathbf{b}} \times \underline{\mathbf{b}})$
- Hence equation can be re-written

$$(\underline{\mathbf{r}} - \underline{\mathbf{a}}) \times \underline{\mathbf{b}} = 0$$

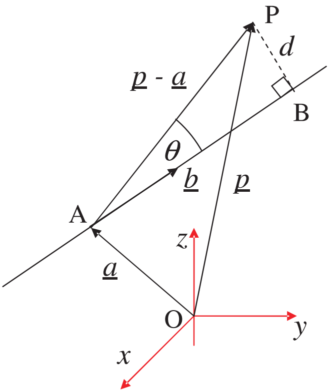
4.1.2 From Cartesian to vector equation

Example

- Start from Cartesian equation:
  $2x - 4 = y + 6 = 3z + 9$
  $\rightarrow$ set this $\equiv \lambda$
- Rewrite this as : $x = \frac{4+\lambda}{2}$, $y = -6 + \lambda$, $z = \frac{-9+\lambda}{3}$
- To get in form $\underline{\mathbf{r}} = \underline{\mathbf{a}} + \lambda \underline{\mathbf{b}}$
  - Get point on line by setting $\lambda = 0$: $\underline{\mathbf{a}} = (2, -6, -3)$;
  - Get vector along line by inspection : $\underline{\mathbf{b}} = (\frac{1}{2}, 1, \frac{1}{3})$
    (or alternatively $\underline{\mathbf{b}} = (3, 6, 2)$)
  - $\underline{\mathbf{r}} = (2, -6, -3) + \lambda(3, 6, 2)$

4.2 Distance from a point to a line

- Line is given by $\underline{\mathbf{r}} = \underline{\mathbf{a}} + \lambda \underline{\mathbf{b}}$
- Define point *P* which has position vector $\underline{\mathbf{p}}$. We want to find the *minimum distance* of point *P* to the line.
- Vector $(\underline{\mathbf{p}} - \underline{\mathbf{a}})$ connects *A* to *P*
- The minimum distance, *d*, from *P* to the line is when angle *ABP* is a right angle.
- From geometry: $d = |\underline{\mathbf{p}} - \underline{\mathbf{a}}| \sin\theta$
- *d* is therefore the magnitude of the vector product $(\underline{\mathbf{p}} - \underline{\mathbf{a}}) \times \underline{\mathbf{b}}/|\underline{\mathbf{b}}|$
- Hence $d = |(\underline{\mathbf{p}} - \underline{\mathbf{a}}) \times \hat{\underline{\mathbf{b}}}|$



4.2.1 Example: minimum distance of a point to a line

Two points $(1, 2, 2)$ and $(3, 0, 5)$ lie on a line. Find the minimum distance of the point $(-2, 3, 1)$ to the line.

Solution:

Line given by $\underline{\mathbf{r}} = \underline{\mathbf{a}} + \lambda \underline{\mathbf{b}}$

- $\underline{\mathbf{a}} = (1, 2, 2)$, $\underline{\mathbf{b}} = (3 - 1, 0 - 2, 5 - 2) = (2, -2, 3)$
- $\hat{\underline{\mathbf{b}}} = (2, -2, 3)/\sqrt{(2^2 + 2^2 + 3^2)} = (2, -2, 3)/\sqrt{(17)}$
- $\underline{\mathbf{p}} = (-2, 3, 1)$
- $\underline{\mathbf{p}} - \underline{\mathbf{a}} = (-2 - 1, 3 - 2, 1 - 2) = (-3, 1, -1)$

$$(\underline{\mathbf{p}} - \underline{\mathbf{a}}) \times \hat{\underline{\mathbf{b}}} = 1/\sqrt{(17)} \times \begin{vmatrix} \underline{\mathbf{i}} & \underline{\mathbf{j}} & \underline{\mathbf{k}} \\ -3 & 1 & -1 \\ 2 & -2 & 3 \end{vmatrix} \qquad (1)$$

$$= 1/\sqrt{(17)} \times (\underline{\mathbf{i}}(1 \times 3 - 1 \times 2) - \underline{\mathbf{j}}(-3 \times 3 + 1 \times 2) + \underline{\mathbf{k}}(3 \times 2 - 1 \times 2))$$

- Minimum distance $= 1/\sqrt{(17)} \times |\underline{\mathbf{i}} + 7\underline{\mathbf{j}} + 4\underline{\mathbf{k}}|$
  $= \sqrt{(1 + 49 + 16)}/\sqrt{(17)} = \sqrt{(66/17)}$ QED

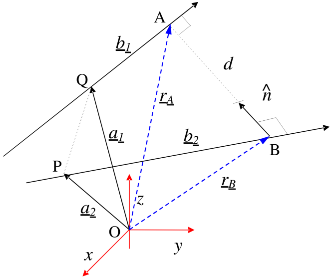
4.3 Minimum distance from a line to a line

- Consider 2 lines in 3D

  $\underline{\mathbf{r_1}} = \underline{\mathbf{a_1}} + \lambda_1 \underline{\mathbf{b_1}}, \quad \underline{\mathbf{r_2}} = \underline{\mathbf{a_2}} + \lambda_2 \underline{\mathbf{b_2}}$

- The shortest distance is represented by the vector perpendicular to both lines

- The unit vector normal to both lines is: $\quad \underline{\hat{\mathbf{n}}} = \frac{\underline{\mathbf{b_1}} \times \underline{\mathbf{b_2}}}{|\underline{\mathbf{b_1}} \times \underline{\mathbf{b_2}}|}$



  - Now let $\underline{\mathbf{r_A}} (= \underline{\mathbf{a_1}} + \lambda_A \underline{\mathbf{b_1}})$ and $\underline{\mathbf{r_B}} (= \underline{\mathbf{a_2}} + \lambda_B \underline{\mathbf{b_2}})$ be the vectors to the points on the lines which correspond to the shortest distance.
  - Vector of shortest distance is then $\quad |\underline{\mathbf{d}}| \, \underline{\hat{\mathbf{n}}} = \underline{\mathbf{r_A}} - \underline{\mathbf{r_B}}$
  - Dot product both sides with vector $\underline{\hat{\mathbf{n}}}$:

    $|\underline{\mathbf{d}}| = (\underline{\mathbf{a_1}} + \lambda_A \underline{\mathbf{b_1}} - \underline{\mathbf{a_2}} - \lambda_B \underline{\mathbf{b_2}}) . \frac{\underline{\mathbf{b_1}} \times \underline{\mathbf{b_2}}}{|\underline{\mathbf{b_1}} \times \underline{\mathbf{b_2}}|}$
  - But $\underline{\mathbf{b_1}}.(\underline{\mathbf{b_1}} \times \underline{\mathbf{b_2}}) = \underline{\mathbf{b_2}}.(\underline{\mathbf{b_1}} \times \underline{\mathbf{b_2}}) = 0$

Hence $|\underline{\mathbf{d}}| = (\underline{\mathbf{a_1}} - \underline{\mathbf{a_2}}).\underline{\hat{\mathbf{n}}} = (\underline{\mathbf{a_1}} - \underline{\mathbf{a_2}}). \frac{\underline{\mathbf{b_1}} \times \underline{\mathbf{b_2}}}{|\underline{\mathbf{b_1}} \times \underline{\mathbf{b_2}}|}$

4.3.1 Example: minimum distance between 2 lines

Consider two lines:

$\underline{\mathbf{r_1}} = (-3, 2, 1) + \lambda_1(4, 2, 2)$; $\quad \underline{\mathbf{r_2}} = (-5, 0, 5) + \lambda_2(3, 2, -1)$

Calculate the minimum distance between the 2 lines.

Solution:

- $\underline{\mathbf{a_1}} = (-3, 2, 1)$, $\underline{\mathbf{b_1}} = (4, 2, 2)$, $\underline{\mathbf{a_2}} = (-5, 0, 5)$, $\underline{\mathbf{b_2}} = (3, 2, -1)$
- $|\underline{\mathbf{d}}| = (\underline{\mathbf{a_1}} - \underline{\mathbf{a_2}}) \, . \, \underline{\hat{\mathbf{n}}} \rightarrow$ Vector defining line of closest distance $\underline{\mathbf{n}}$

$$\underline{\mathbf{n}} = \underline{\mathbf{b_1}} \times \underline{\mathbf{b_2}} = \begin{vmatrix} \underline{\mathbf{i}} & \underline{\mathbf{j}} & \underline{\mathbf{k}} \\ 4 & 2 & 2 \\ 3 & 2 & -1 \end{vmatrix} \quad (2)$$

$= (\underline{\mathbf{i}}(-2-4) - \underline{\mathbf{j}}(-4-6) + \underline{\mathbf{k}}(8-6)) = (-6, 10, 2)$

- Hence $\underline{\hat{\mathbf{n}}} = (-6, 10, 2)/\sqrt{(6^2 + 10^2 + 2^2)}$
- $(\underline{\mathbf{a_1}} - \underline{\mathbf{a_2}}) = (-3+5, 2-0, 1-5) = (2, 2, -4)$
- Minimum distance $|\underline{\mathbf{d}}| = (\underline{\mathbf{a_1}} - \underline{\mathbf{a_2}}) \, . \, \underline{\hat{\mathbf{n}}}$
  $= ((2, 2, -4).(-6, 10, 2))/\sqrt{(140)} = (-12 + 20 - 8)/\sqrt{(140)} = 0$
- Therefore in this case the lines intersect each other!

4.3.2 Angle between the crossing lines

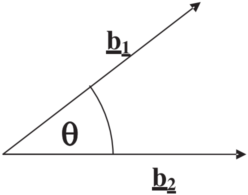
- $\underline{\mathbf{b_1}}.\underline{\mathbf{b_2}} = |\underline{\mathbf{b_1}}|\ |\underline{\mathbf{b_2}}| \cos\theta$

  $\underline{\mathbf{b_1}} = (4, 2, 2), \quad \underline{\mathbf{b_2}} = (3, 2, -1)$

- $\cos\theta = \frac{(4,2,2).(3,2,-1)}{\sqrt(4^2+2^2+2^2)\sqrt(3^2+2^2+1^2)}$

  $= \frac{12+4-2}{\sqrt(24)\sqrt(14)}$

  $= \frac{14}{\sqrt(24)\sqrt(14)} = \sqrt(\frac{14}{24}) = \sqrt(\frac{7}{12}).$



4.3.3 Finding the coordinate at which lines cross

- They cross when $\underline{\mathbf{r_A}} = \underline{\mathbf{r_B}} = \underline{\mathbf{a_1}} + \lambda_A \underline{\mathbf{b_1}} = \underline{\mathbf{a_2}} + \lambda_B \underline{\mathbf{b_2}}$
- Hence $\underline{\mathbf{a_1}} - \underline{\mathbf{a_2}} = \lambda_B \underline{\mathbf{b_2}} - \lambda_A \underline{\mathbf{b_1}}$
- Cross product with $\underline{\mathbf{b_1}}$
  $\underline{\mathbf{b_1}} \times (\underline{\mathbf{a_1}} - \underline{\mathbf{a_2}}) = \lambda_B \underline{\mathbf{b_1}} \times \underline{\mathbf{b_2}}$

$$\Rightarrow \quad \underline{\mathbf{b_1}} \times (\underline{\mathbf{a_1}} - \underline{\mathbf{a_2}}) = \begin{vmatrix} \underline{\mathbf{i}} & \underline{\mathbf{j}} & \underline{\mathbf{k}} \\ 4 & 2 & 2 \\ 2 & 2 & -4 \end{vmatrix} \qquad (3)$$

  $= (-8 - 4, 4 + 16, 8 - 4) \; = \; (-12, 20, 4) = \lambda_B \times (-6, 10, 2)$.
  Clearly $\lambda_B = 2$ by inspection.
- Intersection at $\underline{\mathbf{r_B}} = \underline{\mathbf{a_2}} + \lambda_B \underline{\mathbf{b_2}}$
  $= (-5, 0, 5) + 2 \times (3, 2, -1) \; = \; (1, 4, 3)$.