Classical Mechanics

LECTURE 3:

DIMENSIONAL ANALYSIS & NEWTON'S LAWS

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MT 2016

OUTLINE : 3. DIMENSIONAL ANALYSIS & NEWTON'S LAWS

3.1 Dimensional analysis

- A useful method for determining the units of a variable in an equation
- Useful for checking the correctness of an equation which you have derived after some algebraic manipulation. Dimensions need to be correct !
- Determining the form of an equation itself

Most physical quantities can be expressed in terms of combinations of basic dimensions. These are certainly not unique :

- mass (M)
- length (L)
- time (T)
- electric charge (Q)
- temperature (θ)

Note: The term "dimension" is not quite the same as "unit", but obviously closely related.

Quantity	Unit		Dimension
Frequency	Hertz (Hz)	= (cycles) s^{-1}	T^{-1}
Force	Newton (N)	= kg m s^{-2}	MLT^{-2}
Energy	Joule (J)	= N m = kg m$^2 s^{-2}$	ML^2T^{-2}
Power	Watt (W)	= J s^{-1} = kg m$^2 s^{-3}$	ML^2T^{-3}
Current	Ampere (A)	= Cs^{-1}	QT^{-1}
EMF	Volt (V)	= Nm C^{-1} = kg m$^2 s^{-2} C^{-1}$	$\mathrm{ML^2T^{-2}Q^{-1}}$

Dimensional analysis is best illustrated with examples.

3.1.1 The period of a pendulum

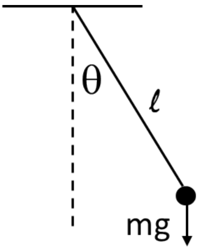
How does the period of a pendulum depend on its length?

- Variables: period P, mass m, length l, acceleration due to gravity g
- Guess the form: let $P = k\, m^a \ell^b g^c$

  (k is a dimensionless constant)
- $T^1 = M^a L^b (LT^{-2})^c = M^a L^{b+c} T^{-2c}$
- Compare terms:

  $a = 0,\ b + c = 0,\ -2c = 1$

  $\rightarrow \quad c = -1/2,\ b = 1/2$



$$P = k\sqrt{\frac{\ell}{g}}$$

We know that $P = 2\pi\sqrt{\frac{\ell}{g}}$: we obtained this form using dimensions and without using equation of motion: IMPRESSIVE !

3.1.2 Kepler's third law

How does the period of an orbiting mass depend on its radius?

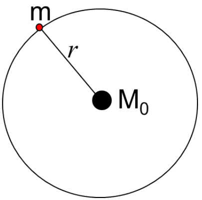
- Variables: period P, central mass M_0, orbit radius r, Gravitational constant G
- Guess the form: let $P = k\, M_0^a r^b G^c$

  (k is a dimensionless constant)
- Dimensions of $G \to (MLT^{-2}).L^2 M^{-2}$
- $T^1 = M^a L^b (M^{-1} L^3 T^{-2})^c$

  $= M^{(a-c)} L^{b+3c} T^{-2c}$
- Compare terms:

  $a - c = 0,\ b + 3c = 0,\ -2c = 1$

  $\to \quad a = -1/2,\ c = -1/2,\ b = 3/2$



- $\frac{GmM_0}{r^2} = \frac{mv^2}{r}$
- $v = \frac{2\pi r}{P}$
- $P^2 = \frac{4\pi^2}{GM_0} r^3$

$P = k\, M_0^{-1/2} r^{3/2} G^{-1/2} \to$ $\boxed{P^2 = \frac{k^2}{GM_0} r^3}$

$\to \quad k^2 = 4\pi^2$

3.1.3 The range of a cannon ball

A cannon ball is fired with V_y upwards and V_x horizontally, assume no air resistance.

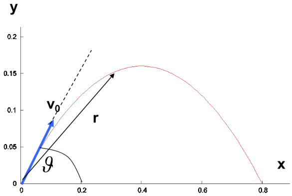
- Variables: V_x, V_y, distance travelled along x (range) R, acceleration due to gravity g
- First with no use of directed length dimensions
- Let $R = kV_x^a V_y^b g^c$.

  (k is a dimensionless constant)
- Dimensionally $L = (L/T)^{a+b}(L/T^2)^c$
- Compare terms:

  $a + b + c = 1$ and $a + b + 2c = 0$, which leaves one exponent undetermined.
- Now use directed length dimensions, then V_x will be dimensioned as L_x/T, V_y as L_y/T, R as L_x and g as L_y/T^2
- The dimensional equation becomes:
  $L_x = (L_x/T)^a (L_y/T)^b (L_y/T^2)^c$

  $\rightarrow a = 1, b = 1$ and $c = -1$.

$$R = k\frac{v_x v_y}{g}$$



- $x = v_x t$
- $y = v_y t - \frac{1}{2}gt^2$

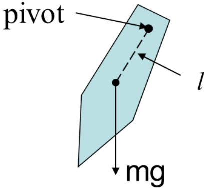
  $= 0$

  $\rightarrow t = \frac{2v_y}{g}$
- $x = \frac{2v_x v_y}{g}$

3.1.4 Example of limitations of the method

- Let $y = f(x_1,\ x_2, \dots x_n)$ where $x_1,\ x_2, \dots x_n$ have independent dimensions
- However in general $y = (x_1^a\, x_2^b \dots x^n)\, \phi(u_1, \dots u_k)$ where u_i are dimensionless variables

Extend to how the period of a *rigid* pendulum depends on length pivot to CM.



- In actual fact $P \equiv P(g, \ell, m, \mathtt{I})$ where $\mathtt{I}$ is the moment of inertia
- $[\mathtt{I}] = ML^2 \ \rightarrow$ can define $u = \frac{\mathtt{I}}{m\ell^2}$

$$T = \sqrt{\frac{\ell}{g}}\,\phi(u)$$

$$T = 2\pi\sqrt{\frac{\mathtt{I}}{mg\ell}}$$

i.e. Equation is not reproduced

3.2 Newton's Laws of motion

- NI: Every body continues in a state of rest or in uniform motion (constant velocity in straight line) unless acted upon by an external force.
- NII: The rate of change of momentum is equal to the applied force; where the momentum is defined as the product of mass and velocity ($\underline{\mathbf{p}} = m\underline{\mathbf{v}}$). [i.e. the applied force $\underline{\mathbf{F}}$ on a body is equal to its mass m multiplied by its acceleration $\underline{\mathbf{a}}$.]
- NIII: When one body exerts a force on a second body, the second body simultaneously exerts a force equal in magnitude and opposite in direction on the first body [i.e. action and reaction forces are equal in magnitude and opposite in direction.]

3.3 Frames of reference

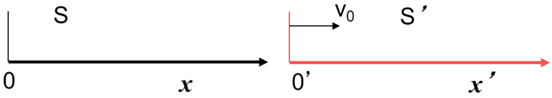
- A frame of reference is an environment which is used to observe an event or the motion of a particle.
- A coordinate system is associated with the frame to observe the event (eg the body's location over time).
- The observer is equipped with measuring tools (eg rulers and clocks) to measure the positions and times of events.
- In classical mechanics, time intervals between events is the same in all reference frames (time is absolute).
- In relativity, we will need to use space-time frames.
- A reference frame in which NI is satisfied is called an *inertial reference frame*.

Inertial reference frames

A frame in which Newton's first law is satisfied:

- Deep space
- The Earth? [Only in circumstances where we can ignore gravity & the spin of the Earth.]

Principle of Relativity : The laws of Physics are the same in all inertial frames of reference.

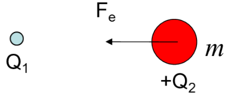


At $t = 0, x = 0, x' = 0$ and S and S' are coincident.

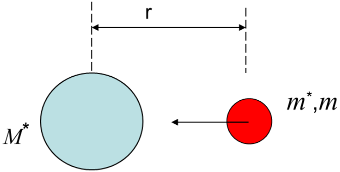
Galilean Transformation of coordinates:

- $x' = x - v_0 t, \quad y' = y, \quad z' = z, \quad t' = t$
- Velocity of a body v in S; velocity measured in S' $\quad v' = v - v_0$
- Acceleration measured in S' $\quad a' = a$
- Hence $F' = F$ (consistent with the principle of relativity)

3.4 The Principle of Equivalence



$$F_e = ma \qquad \left(F_e = k \frac{Q_1 Q_2}{r^2} \right)$$



$$F_g = ma \qquad \left(F_g = G \frac{M^* m^*}{r^2} \right)$$

- The Principle of Equivalence dictates that $m = m^*$.
- Inertial mass = Gravitational mass
- This may seem obvious, but it was not an original postulate of Newton