Classical Mechanics

LECTURE 3:
DIMENSIONAL ANALYSIS & NEWTON’S LAWS

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OUTLINE: 3. DIMENSIONAL ANALYSIS & NEWTON’S LAWS

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3.1 Dimensional analysis

- A useful method for determining the units of a variable in an equation
- Useful for checking the correctness of an equation which you have derived after some algebraic manipulation. Dimensions need to be correct!
- Determining the form of an equation itself

Most physical quantities can be expressed in terms of combinations of basic dimensions. These are certainly not unique:

- mass (M)
- length (L)
- time (T)
- electric charge (Q)
- temperature (θ)
Note: The term "dimension" is not quite the same as "unit", but obviously closely related.

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Unit</th>
<th>Dimension</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency</td>
<td>Hertz (Hz) = (cycles) $s^{-1}$</td>
<td>$T^{-1}$</td>
</tr>
<tr>
<td>Force</td>
<td>Newton (N) = kg m $s^{-2}$</td>
<td>$MLT^{-2}$</td>
</tr>
<tr>
<td>Energy</td>
<td>Joule (J) = N m = kg m$^2$s$^{-2}$</td>
<td>$ML^2T^{-2}$</td>
</tr>
<tr>
<td>Power</td>
<td>Watt (W) = J $s^{-1}$ = kg m$^2$s$^{-3}$</td>
<td>$ML^2T^{-3}$</td>
</tr>
<tr>
<td>Current</td>
<td>Ampere (A) = Cs$^{-1}$</td>
<td>QT$^{-1}$</td>
</tr>
<tr>
<td>EMF</td>
<td>Volt (V) = Nm C$^{-1}$ = kg m$^2$s$^{-2}$C$^{-1}$</td>
<td>ML$^2$T$^{-2}$Q$^{-1}$</td>
</tr>
</tbody>
</table>

Dimensional analysis is best illustrated with examples.
3.1.1 The period of a pendulum

How does the period of a pendulum depend on its length?

- Variables: period $P$, mass $m$, length $l$, acceleration due to gravity $g$

- Guess the form: let $P = k m^a l^b g^c$

  (k is a dimensionless constant)

- $T^1 = M^a L^b (LT^{-2})^c = M^a L^{b+c} T^{-2c}$

- Compare terms:

  $a = 0$, $b + c = 0$, $-2c = 1$

  $\rightarrow c = -1/2$, $b = 1/2$

\[ P = k \sqrt{\frac{l}{g}} \]

We know that $P = 2\pi \sqrt{\frac{l}{g}}$: we obtained this form using dimensions and without using equation of motion: IMPRESSIVE!
3.1.2 Kepler’s third law

How does the period of an orbiting mass depend on its radius?

- Variables: period $P$, central mass $M_0$, orbit radius $r$, Gravitational constant $G$

- Guess the form: let $P = k M_0^a r^b G^c$
  (k is a dimensionless constant)

- Dimensions of $G \rightarrow (MLT^{-2}) \cdot L^2 M^{-2}$

- $T^1 = M^a L^b (M^{-1} L^3 T^{-2})^c$
  $= M^{(a-c)} L^{b+3c} T^{-2c}$

- Compare terms:
  $a - c = 0\), $b + 3c = 0\), $-2c = 1$
  $\rightarrow a = -1/2\), $c = -1/2\), $b = 3/2$

$P = k M_0^{-1/2} r^{3/2} G^{-1/2}$ $\rightarrow$ $P^2 = \frac{k^2}{GM_0} r^3$

$\rightarrow k^2 = 4\pi^2$
3.1.3 The range of a cannon ball

A cannon ball is fired with $V_y$ upwards and $V_x$ horizontally, assume no air resistance.

▶ Variables: $V_x$, $V_y$, distance travelled along $x$ (range) $R$, acceleration due to gravity $g$

▶ First with no use of directed length dimensions

▶ Let $R = kV_x^a V_y^b g^c$.
   (k is a dimensionless constant)

▶ Dimensionally $L = (L/T)^{a+b}(L/T^2)^c$

▶ Compare terms:
   $a + b + c = 1$ and $a + b + 2c = 0$, which leaves one exponent undetermined.

▶ Now use directed length dimensions, then $V_x$ will be dimensioned as $L_x/T$, $V_y$ as $L_y/T$, $R$ as $L_x$ and $g$ as $L_y/T^2$

▶ The dimensional equation becomes:
   $L_x = (L_x/T)^a (L_y/T)^b (L_y/T^2)^c$
   $\rightarrow a = 1, b = 1$ and $c = -1$.

$$R = k \frac{V_x V_y}{g}$$
3.1.4 Example of limitations of the method

- Let \( y = f(x_1, x_2, \ldots x_n) \) where \( x_1, x_2, \ldots x_n \) have independent dimensions

- However in general \( y = (x_1^a x_2^b \ldots x^n) \phi(u_1, \ldots u_k) \) where \( u_i \) are dimensionless variables

Extend to how the period of a rigid pendulum depends on length pivot to CM.

- In actual fact \( P \equiv P(g, \ell, m, I) \) where \( I \) is the moment of inertia

- \([I] = ML^2 \rightarrow \) can define \( u = \frac{I}{m\ell^2} \)

\[
T = \sqrt{\frac{\ell}{g}} \phi(u)
\]

i.e. Equation is not reproduced

\[
T = 2\pi \sqrt{\frac{I}{mgl}}
\]
3.2 Newton’s Laws of motion

- NI: Every body continues in a state of rest or in uniform motion (constant velocity in straight line) unless acted upon by an external force.

- NII: The rate of change of momentum is equal to the applied force; where the momentum is defined as the product of mass and velocity \((p = mv)\). [i.e. the applied force \(F\) on a body is equal to its mass \(m\) multiplied by its acceleration \(a\).]

- NIII: When one body exerts a force on a second body, the second body simultaneously exerts a force equal in magnitude and opposite in direction on the first body [i.e. action and reaction forces are equal in magnitude and opposite in direction.]
3.3 Frames of reference

- A frame of reference is an environment which is used to observe an event or the motion of a particle.

- A coordinate system is associated with the frame to observe the event (e.g., the body’s location over time).

- The observer is equipped with measuring tools (e.g., rulers and clocks) to measure the positions and times of events.

- In classical mechanics, time intervals between events is the same in all reference frames (time is absolute).

- In relativity, we will need to use space-time frames.

- A reference frame in which NI is satisfied is called an *inertial reference frame*. 
Inertial reference frames

A frame in which Newton’s first law is satisfied:

- Deep space
- The Earth? [Only in circumstances where we can ignore gravity & the spin of the Earth.]

**Principle of Relativity**: The laws of Physics are the same in all inertial frames of reference.

Galilean Transformation of coordinates:

- \( x' = x - v_0 t \), \( y' = y \), \( z' = z \), \( t' = t \)
- Velocity of a body \( v \) in \( S \); velocity measured in \( S' \) \( v' = v - v_0 \)
- Acceleration measured in \( S' \) \( a' = a \)
- Hence \( F' = F \) (consistent with the principle of relativity)
3.4 The Principle of Equivalence

The Principle of Equivalence dictates that $m = m^*$. 

- Inertial mass = Gravitational mass 
- This may seem obvious, but it was not an original postulate of Newton