LECTURE 3:

MORE ON VECTOR PRODUCTS

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Outline: 3. VECTOR PRODUCTS AND GEOMETRY

3.1 Scalar Triple Product

- The scalar triple product, $\underline{\mathbf{a}}.(\underline{\mathbf{b}} \times \underline{\mathbf{c}})$, is the scalar product of the vector $\underline{\mathbf{a}}$ with the cross products of vectors $(\underline{\mathbf{b}} \times \underline{\mathbf{c}})$.
- The result is a scalar.
- Scalar triple product is also written $[\underline{\mathbf{a}}, \underline{\mathbf{b}}, \underline{\mathbf{c}}]$.
- Scalar triple product in component form :

$$\underline{\mathbf{a}}.(\underline{\mathbf{b}} \times \underline{\mathbf{c}}) = \underline{\mathbf{a}}. \begin{vmatrix} \underline{\mathbf{i}} & \underline{\mathbf{j}} & \underline{\mathbf{k}} \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} \quad (1)$$

$$= (a_x\underline{\mathbf{i}}+a_y\underline{\mathbf{j}}+a_z\underline{\mathbf{k}}).((b_y c_z - b_z c_y)\underline{\mathbf{i}} - (b_x c_z - b_z c_x)\underline{\mathbf{j}} + (b_x c_y - b_y c_x)\underline{\mathbf{k}})$$

$$= a_x(b_y c_z - b_z c_y) - a_y(b_x c_z - b_z c_x) + a_z(b_x c_y - b_y c_x)$$

$$\text{In matrix (determinant) form : } \underline{\mathbf{a}}.(\underline{\mathbf{b}} \times \underline{\mathbf{c}}) = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} \quad (2)$$

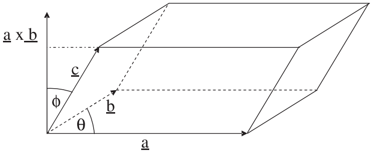
3.1.1 Properties of scalar triple product

- It is obvious that $\underline{\mathbf{a}}.(\underline{\mathbf{b}} \times \underline{\mathbf{c}}) = (\underline{\mathbf{b}} \times \underline{\mathbf{c}}).\underline{\mathbf{a}}$
- *Cyclic* permutations of $\underline{\mathbf{a}}$, $\underline{\mathbf{b}}$ and $\underline{\mathbf{c}}$ leaves the triple scalar product unaltered:
  $\underline{\mathbf{a}}.(\underline{\mathbf{b}} \times \underline{\mathbf{c}}) = \underline{\mathbf{c}}.(\underline{\mathbf{a}} \times \underline{\mathbf{b}}) = \underline{\mathbf{b}}.(\underline{\mathbf{c}} \times \underline{\mathbf{a}})$
  (easily derived by working in components).
- *Non-cyclic* permutations change sign:
  $[\underline{\mathbf{a}}, \underline{\mathbf{b}}, \underline{\mathbf{c}}] = [\underline{\mathbf{c}}, \underline{\mathbf{a}}, \underline{\mathbf{b}}] = [\underline{\mathbf{b}}, \underline{\mathbf{c}}, \underline{\mathbf{a}}] =$
  $-[\underline{\mathbf{a}}, \underline{\mathbf{c}}, \underline{\mathbf{b}}] = -[\underline{\mathbf{c}}, \underline{\mathbf{b}}, \underline{\mathbf{a}}] = -[\underline{\mathbf{b}}, \underline{\mathbf{a}}, \underline{\mathbf{c}}]$
- The scalar triple product is zero if any two vectors are parallel.
- The scalar triple product is zero if the three vectors are coplanar (lie in the same plane).

3.1.2 Geometrical interpretation

The triple scalar product can be interpreted as the volume of a parallelepiped:

- [Volume] = [Area of base] $\times$ [Vertical height of parallelepiped]
- [Area of base] = $|\underline{\mathbf{a}} \times \underline{\mathbf{b}}|$ (vector direction is perpendicular to the base)
- [Vertical height] $= |\underline{\mathbf{c}}| \cos\phi = \underline{\mathbf{c}} \cdot (\frac{\underline{\mathbf{a}}\times\underline{\mathbf{b}}}{|\underline{\mathbf{a}}\times\underline{\mathbf{b}}|})$



- Hence [Volume] $= |\underline{\mathbf{a}} \times \underline{\mathbf{b}}| \left(\underline{\mathbf{c}} \cdot (\frac{\underline{\mathbf{a}}\times\underline{\mathbf{b}}}{|\underline{\mathbf{a}}\times\underline{\mathbf{b}}|})\right) = \underline{\mathbf{c}} \cdot (\underline{\mathbf{a}} \times \underline{\mathbf{b}})$
- Obviously if $\underline{\mathbf{a}}, \underline{\mathbf{b}}$ and $\underline{\mathbf{c}}$ are coplanar, volume = 0.

Example

Calculate the volume of a parallelepiped defined by vectors $(1,1,2),(1,3,2),(-2,1,1)$ from the origin :

- Solution:

$$\text{Volume} = \underline{\mathbf{c}}.(\underline{\mathbf{a}} \times \underline{\mathbf{b}}) = \begin{vmatrix} -2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 3 & 2 \end{vmatrix} \tag{3}$$

$$= -2(1 \times 2 - 3 \times 2) - 1(2 \times 1 - 2 \times 1) + 1(1 \times 3 - 1 \times 1)$$

$$= 8 - 0 + 2$$

$$= 10$$

3.2 Vector Triple Product

- The vector triple vector product, $\underline{\mathbf{a}} \times (\underline{\mathbf{b}} \times \underline{\mathbf{c}})$, is the vector product of the vector $\underline{\mathbf{a}}$ with the cross products of vectors $(\underline{\mathbf{b}} \times \underline{\mathbf{c}})$.
- The result is a vector.
- This is *not* associative. i.e. $\underline{\mathbf{a}} \times (\underline{\mathbf{b}} \times \underline{\mathbf{c}}) \neq (\underline{\mathbf{a}} \times \underline{\mathbf{b}}) \times \underline{\mathbf{c}}$.
- Clearly for $\underline{\mathbf{a}} \times (\underline{\mathbf{b}} \times \underline{\mathbf{c}})$, the vector lies in the plane of $\underline{\mathbf{b}}$ and $\underline{\mathbf{c}}$ and can be expressed in terms of them.

It can be shown:

$$\underline{\mathbf{a}} \times (\underline{\mathbf{b}} \times \underline{\mathbf{c}}) = (\underline{\mathbf{a}}.\underline{\mathbf{c}})\, \underline{\mathbf{b}} - (\underline{\mathbf{a}}.\underline{\mathbf{b}})\, \underline{\mathbf{c}}$$

(partial proof, see over ...).

Partial proof (x-component only):

$$(\underline{\mathbf{b}} \times \underline{\mathbf{c}}) = \begin{vmatrix} \underline{\mathbf{i}} & \underline{\mathbf{j}} & \underline{\mathbf{k}} \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} \tag{4}$$

$$\underline{\mathbf{a}} \times (\underline{\mathbf{b}} \times \underline{\mathbf{c}}) = \begin{vmatrix} \underline{\mathbf{i}} & \underline{\mathbf{j}} & \underline{\mathbf{k}} \\ a_x & a_y & a_z \\ b_y c_z - b_z c_y & -(b_x c_z - b_z c_x) & b_x c_y - b_y c_x \end{vmatrix} \tag{5}$$

- x-component only

  $\underline{\mathbf{i}}: \quad a_y(b_x c_y - b_y c_x) + a_z(b_x c_z - b_z c_x)$

  $= (a_y c_y + a_z c_z).b_x - (a_y b_y + a_z b_z).c_x +$

  $+ \ ((a_x c_x)b_x - (a_x b_x)c_x) \quad \leftarrow [\text{note, add this extra term, sum} = 0]$

  $= \underline{\mathbf{i}} \ ((\underline{\mathbf{a}}.\underline{\mathbf{c}}) \ b_x - (\underline{\mathbf{a}}.\underline{\mathbf{b}}) \ c_x)$ Similarly for $\underline{\mathbf{j}}$ and $\underline{\mathbf{k}}$ components.

- Also easy to show:

  $(\underline{\mathbf{a}} \times \underline{\mathbf{b}}) \times \underline{\mathbf{c}} = (\underline{\mathbf{a}}.\underline{\mathbf{c}}) \ \underline{\mathbf{b}} - (\underline{\mathbf{b}}.\underline{\mathbf{c}}) \ \underline{\mathbf{a}}$.

- Can also show from above expressions:

  $\underline{\mathbf{a}} \times (\underline{\mathbf{b}} \times \underline{\mathbf{c}}) + \underline{\mathbf{b}} \times (\underline{\mathbf{c}} \times \underline{\mathbf{a}}) + \underline{\mathbf{c}} \times (\underline{\mathbf{a}} \times \underline{\mathbf{b}}) = 0$

3.2.1 Lagrange's identity

Another useful identity (can be proved using components)

$$(\underline{\mathbf{a}} \times \underline{\mathbf{b}}).(\underline{\mathbf{c}} \times \underline{\mathbf{d}}) = (\underline{\mathbf{a}}.\underline{\mathbf{c}})(\underline{\mathbf{b}}.\underline{\mathbf{d}}) - (\underline{\mathbf{a}}.\underline{\mathbf{d}})(\underline{\mathbf{b}}.\underline{\mathbf{c}})$$

Or alternatively: can be proved using identities of scalar and vector triple products:

- $(\underline{\mathbf{a}} \times \underline{\mathbf{b}}) \cdot (\underline{\mathbf{c}} \times \underline{\mathbf{d}}) = \underline{\mathbf{d}} \cdot ((\underline{\mathbf{a}} \times \underline{\mathbf{b}}) \times \underline{\mathbf{c}}) = \underline{\mathbf{c}} \cdot (\underline{\mathbf{d}} \times (\underline{\mathbf{a}} \times \underline{\mathbf{b}}))$
  (Using properties of scalar triple product)
- $= \underline{\mathbf{c}} \cdot ((\underline{\mathbf{d}} \cdot \underline{\mathbf{b}})\, \underline{\mathbf{a}} - (\underline{\mathbf{d}} \cdot \underline{\mathbf{a}})\, \underline{\mathbf{b}})$
  (Using identity of vector product)
- $= (\underline{\mathbf{a}} \cdot \underline{\mathbf{c}})(\underline{\mathbf{b}} \cdot \underline{\mathbf{d}}) - (\underline{\mathbf{a}} \cdot \underline{\mathbf{d}})(\underline{\mathbf{b}} \cdot \underline{\mathbf{c}})$
  (Rearranging)

3.3 Generating orthogonal axes

Orthogonal axes can be constructed from cross product of two general vectors

Prescription:

- i) Start from general vectors $\underline{\mathbf{a}}$ and $\underline{\mathbf{b}}$,
- ii) Choose vector $\underline{\mathbf{a}}$ as the direction of the *x*-axis
- iii) The direction of the *y*-axis is then given by $\underline{\mathbf{a}} \times \underline{\mathbf{b}}$
- iv) The direction of the *z*-axis is then simply given by $\underline{\mathbf{a}} \times (\underline{\mathbf{a}} \times \underline{\mathbf{b}})$.