Classical Mechanics

LECTURE 29:

FINAL LAGRANGIAN EXAMPLES

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OUTLINE : 29. FINAL LAGRANGIAN EXAMPLES

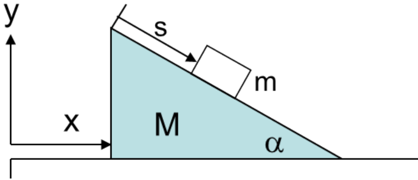
29.1 Re-examine the sliding blocks using E-L

29.2 Normal modes of coupled identical springs

29.3 Final example: a rotating coordinate system

29.1 Re-examine the sliding blocks using E-L

A block of mass m slides on a frictionless inclined plane of mass M, which itself rests on a horizontal frictionless surface. Find the acceleration of the inclined plane.



- Reduce the problem to two generalized coordinates, x and s
- Motion of the inclined plane :

  $T_M = \frac{1}{2} M \dot{x}^2$
- Motion of the block :

  $T_m = \frac{1}{2} m(\dot{x}'^2 + \dot{y}'^2)$ where
- $x' = x + s\cos\alpha$; $y' = -s\sin\alpha$
- $\dot{x}' = \dot{x} + \dot{s}\cos\alpha$; $\dot{y}' = -\dot{s}\sin\alpha$
  - $T_m = \frac{1}{2} m\left[(\dot{x} + \dot{s}\cos\alpha)^2 + \frac{1}{2} m(\dot{s}\sin\alpha)^2\right]$
  - $T = T_m + T_M = \frac{1}{2}(m + M)\dot{x}^2 + \frac{1}{2} m\left(\dot{s}^2 + 2\dot{x}\dot{s}\cos\alpha\right)$
  - $U = -m\,g\,s\sin\,\alpha$

Sliding blocks, continued

- Lagrangian $\quad L = T - U$
- $L = \frac{1}{2}(m+M)\dot{x}^2 + \frac{1}{2}m\left(\dot{s}^2 + 2\dot{x}\dot{s}\cos\alpha\right) + mgs\sin\alpha$
- 2 generalized coordinates $\rightarrow$ x and s
- The E-L equation $\frac{\partial L}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = 0$
- E-L for x : $\frac{\partial L}{\partial x} = 0$; $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{d}{dt}\left[(m+M)\dot{x} + m\dot{s}\cos\alpha\right]$

  $\rightarrow \quad (m+M)\ddot{x} + m\ddot{s}\cos\alpha = 0 \qquad (1)$
- E-L for s : $\frac{\partial L}{\partial s} = mg\sin\alpha$; $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{s}}\right) = \frac{d}{dt}\left[m\left(\dot{s} + \dot{x}\cos\alpha\right)\right]$

  $\rightarrow \quad \ddot{s} + \ddot{x}\cos\alpha = g\sin\alpha \qquad (2)$
- Rearranging (1) & (2)

$$\ddot{x} = -g\,\frac{\sin\alpha\cos\alpha}{\sin^2\alpha + M/m} \quad ; \quad \ddot{s} = g\,\frac{\sin\alpha(1+M/m)}{\sin^2\alpha + M/m}$$

- From (1) $\qquad M\dot{x} + m(\dot{x} + \dot{s}\cos\alpha) = \text{const.}$

  $\rightarrow \quad M\dot{x} + m\dot{x}' = \text{const.}$ Conservation of momentum.

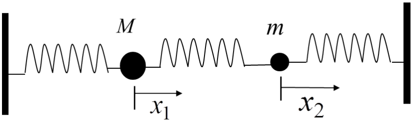
29.2 Normal modes of coupled identical springs

Coupled identical springs mounted horizontally. x_1 and x_2 measure displacements from the respective equilibrium positions. Assume the springs are unstretched at equilibrium.

- The problem has two generalized coordinates, x_1 and x_2

  $T = \frac{1}{2}M\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2$

  $U = \frac{1}{2}kx_1^2 + \frac{1}{2}kx_2^2 + \frac{1}{2}k\,(x_2 - x_1)^2$



- $L = T - U$
  - E-L equation for x_1 : $\frac{\partial L}{\partial x_1} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) = 0$
  - $\frac{\partial L}{\partial x_1} = -kx_1 + k(x_2 - x_1) = k(x_2 - 2x_1)$; $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) = M\ddot{x}_1$

    $\rightarrow \quad M\ddot{x}_1 = k(x_2 - 2x_1)$; $m\ddot{x}_2 = k(x_1 - 2x_2)$
  - $\begin{pmatrix} M\,\ddot{x}_1 \\ m\,\ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} 2k & -k \\ -k & 2k \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

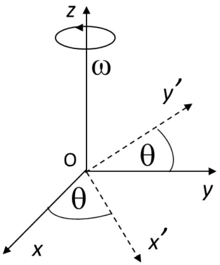
Coupled identical springs, continued

- $$\begin{pmatrix} M\ddot{x}_1 \\ m\ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} 2k & -k \\ -k & 2k \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad (1)$$
- SHM solutions $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} exp(i\,\omega\,t)$
- Substitute into (1)

$$-\omega^2 \underbrace{\begin{pmatrix} M & 0 \\ 0 & m \end{pmatrix}}_{\underline{\mathbf{M}}} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = -\underbrace{\begin{pmatrix} 2k & -k \\ -k & 2k \end{pmatrix}}_{\underline{\mathbf{K}}} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

- Putting $\omega^2 = \lambda \quad \rightarrow \qquad \underline{\mathbf{M}}^{-1}\underline{\mathbf{K}}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \lambda \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$
- Eigenvalue equation; homogeneous solutions
- Etc etc ....

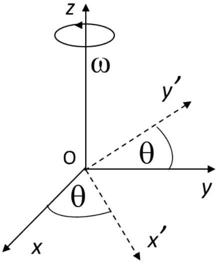
29.3 Final example: a rotating coordinate system



- Lagrangian of a free particle :
  $L = \frac{1}{2} m \dot{\underline{\mathbf{r}}}^2$, $\underline{\mathbf{r}} = (x, y, z)$ (with $U = 0$)
- Measure the motion w.r.t. a coordinate system rotating with angular velocity $\underline{\omega} = (0, 0, \omega)$ about the z axis.
- $\underline{\mathbf{r}}' = (x', y', z')$ are coordinates in the rotating system
- $$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\omega t & \sin\omega t & 0 \\ -\sin\omega t & \cos\omega t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$
  - Take the inverse :
    $$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos\omega t & -\sin\omega t & 0 \\ \sin\omega t & \cos\omega t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$
  - Substitute these expressions into the Lagrangian above $\rightarrow$ find L in terms of the rotating coordinates

A rotating coordinate system, continued

- $L = \frac{1}{2}m\left[(\dot{x}' - \omega\, y')^2 + (\dot{y}' + \omega\, x')^2 + \dot{z}'^2\right]$
  $= \frac{1}{2}m(\dot{\underline{\mathbf{r}}}' + \underline{\omega} \times \underline{\mathbf{r}}')^2$ in the general case
- In this rotating frame, we can use Lagrange equations to derive the equations of motion. Taking derivatives, we have



- $\frac{\partial L}{\partial \underline{\mathbf{r}}'} = m\left[\dot{\underline{\mathbf{r}}}' \times \underline{\omega} - \underline{\omega} \times (\underline{\omega} \times \underline{\mathbf{r}}')\right]$

  where $\frac{\partial}{\partial \underline{\mathbf{r}}'} = \left(\frac{\partial}{\partial x'}, \frac{\partial}{\partial y'}, \frac{\partial}{\partial z'}\right)$

  $\rightarrow \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\underline{\mathbf{r}}}'}\right) = m\frac{d}{dt}(\dot{\underline{\mathbf{r}}}' + \underline{\omega} \times \underline{\mathbf{r}}') = m(\ddot{\underline{\mathbf{r}}}' + \underline{\omega} \times \dot{\underline{\mathbf{r}}}')$

  - So the Lagrange equation becomes

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\underline{\mathbf{r}}}'}\right) - \frac{\partial L}{\partial \underline{\mathbf{r}}'} = m[\underbrace{\ddot{\underline{\mathbf{r}}}'}_{\text{radial force}} + \underbrace{\underline{\omega} \times (\underline{\omega} \times \underline{\mathbf{r}}')}_{\text{Centrifugal force}} + \underbrace{2\,\underline{\omega} \times \dot{\underline{\mathbf{r}}}'}_{\text{Coriolis force}}] = 0$$

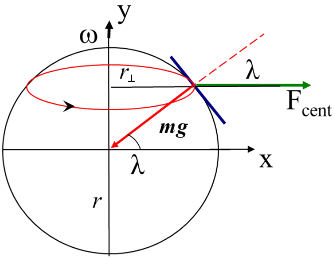
A rotating coordinate system, continued

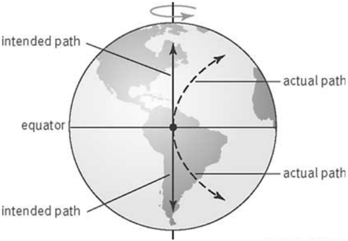
- Centrifugal and Coriolis forces are examples of “fictitious forces” :

  $\rightarrow$ called “fictitious” since they are a consequence of the reference frame, rather than any interaction. The forces do not exist in an inertial frame.

- The centrifugal force $\underline{\mathbf{F}}_{cent} = m\underline{\omega} \times (\underline{\omega} \times \underline{\mathbf{r}}')$ points outwards in the plane perpendicular to $\underline{\omega}$ with magnitude $m\omega^2 |r'_\perp|$ ($\perp$ is the projection perpendicular to $\underline{\omega}$)

- The Coriolis force $\underline{\mathbf{F}}_{cor} = 2m\underline{\omega} \times \dot{\underline{\mathbf{r}}}'$ acts in a direction perpendicular to the rotation axis $\underline{\omega}$ and to the velocity of the body in the rotating frame



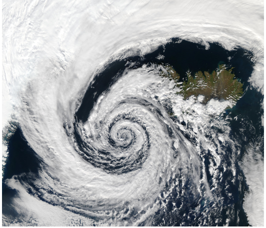


A rotating coordinate system, continued

- Coriolis force responsible for the circulation of oceans and the atmosphere.
- A projectile thrown in the northern hemisphere rotates in a clockwise direction
- A projectile thrown in the southern hemisphere rotates in an anti-clockwise direction.
- For a particle moving along the equator, $\underline{\omega} \perp \underline{\dot{\mathbf{r}}}'$, the Coriolis force tends to zero $\rightarrow$ no effect on the projectile
- The Coriolis force is responsible for the formation of hurricanes. These rotate in different directions in the northern and southern hemisphere. They never form within 500 miles of the equator where the Coriolis force is too weak.

Australia

Iceland



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