Classical Mechanics
LECTURE 28:
HAMILTONIAN MECHANICS, NOETHER'S THEOREM

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HT 2017

OUTLINE : 28. HAMILTONIAN MECHANICS, NOETHER'S THEOREM

28.1 Hamilton mechanics

28.2 The physical significance of the Hamiltonian

28.3 Example: re-visit bead on rotating hoop

28.4 Noether's theorem

28.1 Hamilton mechanics

- Lagrangian mechanics : Allows us to find the equations of motion for a system in terms of an arbitrary set of generalized coordinates
- Now extend the method due to Hamilton
  $\rightarrow$ use of the conjugate (generalized) momenta $p_1, p_2, \cdots, p_n$ replace the generalized velocities $\dot{q}_1, \dot{q}_2, \cdots, \dot{q}_n$
- This has advantages when some of conjugate momenta are constants of the motion and it is well suited to finding conserved quantities
- From before, conjugate momentum : $p_k = \frac{\partial L}{\partial \dot{q}_k}$
  and E-L equation reads for coordinate k : $\dot{p}_k = \frac{\partial L}{\partial q_k}$
  (since E-L is $\dot{p}_k = \frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_k}) = \frac{\partial L}{\partial q_k}$)

The Hamiltonian, continued

- Lagrangian $L = L(q_k, \dot{q}_k, t) \implies$
- $\frac{dL}{dt} = \frac{\partial L}{\partial t} + \sum_k \left(\frac{\partial L}{\partial q_k}\frac{dq_k}{dt} + \frac{\partial L}{\partial \dot{q}_k}\frac{d\dot{q}_k}{dt} \right)$
  $= \frac{\partial L}{\partial t} + \sum_k \left(\frac{\partial L}{\partial q_k}\dot{q}_k + \frac{\partial L}{\partial \dot{q}_k}\ddot{q}_k \right)$

- An aside: use rules of partial differentiation:
- If $f = f(x, y, z)$
- $\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial z}\frac{dz}{dx}$

- Conjugate momentum definition : $p_k = \frac{\partial L}{\partial \dot{q}_k}$, $\dot{p}_k = \frac{\partial L}{\partial q_k}$
- Therefore $\frac{dL}{dt} = \frac{\partial L}{\partial t} + \sum_k \underbrace{(\dot{p}_k \dot{q}_k + p_k \ddot{q}_k)}_{\frac{d}{dt}(p_k \dot{q}_k)}$
- $\frac{d}{dt}\underbrace{(L - \sum_k p_k \dot{q}_k)}_{-H} = \frac{\partial L}{\partial t}$ $\qquad \frac{dH}{dt} = -\frac{\partial L}{\partial t}$
- Define Hamiltonian $\qquad H = \sum_k p_k \dot{q}_k - L$
- If L does not depend *explicitly* on time, H is a constant of motion

28.2 The physical significance of the Hamiltonian

- From before : $H = \sum_k p_k \dot{q}_k - L$
- Where conjugate momentum : $p_k = \frac{\partial L}{\partial \dot{q}_k}$, $\dot{p}_k = \frac{\partial L}{\partial q_k}$
- Take kinetic energy $T = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 \right)$
- $L = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 \right) - U(x, y, z)$
- $H = \sum_k p_k \dot{q}_k - L = \frac{1}{2} m (2\dot{x}.\dot{x} + 2\dot{y}.\dot{y} + 2\dot{z}.\dot{z}) - (T - U)$

  $= 2T - (T - U) = T + U = E \quad \rightarrow$ total energy
- From before $\frac{dH}{dt} = -\frac{\partial L}{\partial t}$

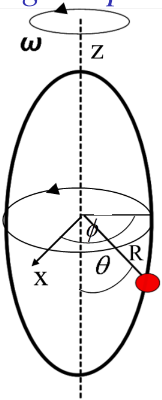
  $\rightarrow$ If L does not depend *explicitly* on time $\frac{dH}{dt} = 0$

  $\rightarrow$ energy is a constant of the motion
- Can show by differentiation :

  Hamilton Equations $\rightarrow$ $\frac{\partial H}{\partial p_k} = \dot{q}_k \;\; ; \;\; \frac{\partial H}{\partial q_k} = -\dot{p}_k$

  If a coordinate does not appear in the Hamiltonian it is *cyclic* or *ignorable*

28.3 Example: re-visit bead on rotating hoop



First take the case of a free (undriven) system

- $L = \frac{1}{2} m (R^2 \dot\theta^2 + R^2 \sin^2\theta\, \dot\phi^2) + mgR\cos\theta$
- $H = \sum_k p_k \dot q_k - L$; $p_k = \frac{\partial L}{\partial \dot q_k}$
- $p_\theta = \frac{\partial L}{\partial \dot\theta} = m R^2 \dot\theta$; $p_\phi = mR^2 \sin^2\theta\, \dot\phi$
- $H = m R^2 \dot\theta^2 + mR^2 \sin^2\theta\, \dot\phi^2 - L$

$$= \frac{1}{2} m \left(R^2 \dot\theta^2 + R^2 \sin^2\theta\, \dot\phi^2 \right) - mgR\cos\theta$$

$$\rightarrow \; H = T + U = E$$

L does not depend explicitly on t,
H, E conserved $\rightarrow$ Hamiltonian gives the total energy

Hamilton Equations : $\dot q_k = \frac{\partial H}{\partial p_k}$; $\dot p_k = -\frac{\partial H}{\partial q_k}$

$$\rightarrow \; \dot p_\phi = -\frac{\partial H}{\partial \phi} = 0 \quad \text{(ignorable)}$$

$\rightarrow \quad p_\phi = mR^2 \sin^2\theta\, \dot\phi = J_z$ = constant of the motion

Example continued

Now consider a DRIVEN system - hoop rotating at constant angular speed ω

- $L = \frac{1}{2} m \left(R^2 \dot{\theta}^2 + R^2 \omega^2 \sin^2 \theta\right) + mgR \cos\theta$
- $H = \sum_k p_k \dot{q}_k - L \;\; ; \;\; p_k = \frac{\partial L}{\partial \dot{q}_k}$
- $p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m R^2 \dot{\theta}$; a single coordinate θ
- $H = m R^2 \dot{\theta}^2 - L$

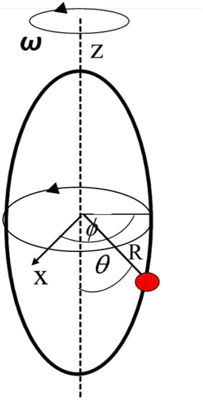
$$= \frac{1}{2} m \left(R^2 \dot{\theta}^2 - R^2 \omega^2 \sin^2 \theta\right) - mgR \cos\theta$$

- $E = \frac{1}{2} m \left(R^2 \dot{\theta}^2 + R^2 \omega^2 \sin^2 \theta\right) - mgR \cos\theta$

Hence $E = H + mR^2 \omega^2 \sin^2 \theta$

$\rightarrow \;\; E \; (= T + U) \neq H$

So what's different ?



- $\frac{dH}{dt} = -\frac{\partial L}{\partial t}$
- H is a constant of the motion, E is not const.

In this case the hoop has been forced to rotate at an angular velocity ω. External energy is being supplied to the system.

28.4 Noether's theorem

The theorem states : *Whenever there is a continuous symmetry of the Lagrangian, there is an associated conservation law.*

- *Symmetry* means a transformation of the generalized coordinates q_k and $\dot{q}_k$ that leaves the value of the Lagrangian unchanged.
- If a Lagrangian does not depend on a coordinate q_k (ie. is cyclic) it is invariant (*symmetric*) under changes $q_k \to q_k + \delta q_k$; the corresponding generalized momentum $p_k = \frac{\partial L}{\partial \dot{q}_k}$ is conserved

1. For a Lagrangian that is symmetric under changes $t \to t + \delta t$, the total energy H is conserved $\to H = \sum_k \frac{\partial L}{\partial \dot{\mathbf{q}}} \dot{\mathbf{q}} - L$
2. For a Lagrangian that is symmetric under changes $r \to r + \delta r$, the linear momentum $\underline{\mathbf{p}}$ is conserved
3. For a Lagrangian that is symmetric under small rotations of angle $\theta \to \theta + \delta\theta$ about an axis $\underline{\hat{\mathbf{n}}}$ such a rotation transforms the Cartesian coordinates by $\underline{\mathbf{r}} \to \underline{\mathbf{r}} + \delta\theta\, \underline{\hat{\mathbf{n}}} \times \underline{\mathbf{r}}$, the conserved quantity is the component of the angular momentum $\underline{\mathbf{J}}$ along the $\underline{\hat{\mathbf{n}}}$ axis