Classical Mechanics
LECTURE 27:
MORE LAGRANGE EXAMPLES

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OUTLINE : 27. MORE LAGRANGE EXAMPLES

27.1 The Lagrangian in various coordinate systems

27.2 Example 1: the rotating bead

27.3 Example 2: bead on rotating hoop

27.1 The Lagrangian in various coordinate systems

- Cartesian coordinates

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U(x, y, z)$$

Already shown that E-L gives

$$m\ddot{x} = -\frac{\partial U}{\partial x}\ ;\ m\ddot{y} = -\frac{\partial U}{\partial y};\ m\ddot{z} = -\frac{\partial U}{\partial z}$$

$$\rightarrow \quad m\,\underline{\ddot{\mathbf{r}}} = -\nabla U$$

- Cylindrical coordinates

$$x = r\cos\phi\ ;\ \dot{x} = \dot{r}\cos\phi - r\sin\phi\,\dot{\phi}$$

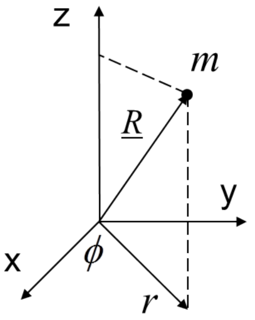
$$y = r\sin\phi\ ;\ \dot{y} = \dot{r}\sin\phi + r\cos\phi\,\dot{\phi}$$

$$z = z\ ;\ \dot{z} = \dot{z}$$

$$T = \frac{1}{2}m\,(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

$$= \frac{1}{2}m[(r^2\cos^2\phi + r^2\sin^2\phi\dot{\phi}^2 - 2r\dot{r}\cos\phi\sin\phi\dot{\phi})]$$
$$+(r^2\sin^2\phi + r^2\cos^2\phi\dot{\phi}^2 + 2r\dot{r}\cos\phi\sin\phi\dot{\phi}) + \dot{z}^2]$$

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\,\dot{\phi}^2 + \dot{z}^2) - U(r, \phi, z)$$



Cylindrical coords

ϕ is cyclic if $U = U(r)$ only

$$\rightarrow \quad p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi}$$

$\rightarrow$ constant angular momentum

The Lagrangian in various coordinate systems continued

- Spherical coordinates

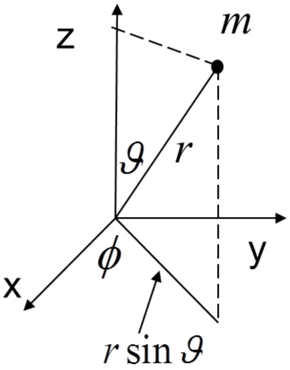
$$x = r\sin\theta\cos\phi\ ;\ \dot{x} = \dot{r}\sin\theta\cos\phi + r\cos\theta\cos\phi\,\dot{\theta} - r\sin\theta\sin\phi\,\dot{\phi}$$
$$y = r\sin\theta\sin\phi\ ;\ \dot{y} = \dot{r}\sin\theta\sin\phi + r\cos\theta\sin\phi\,\dot{\theta} + r\sin\theta\cos\phi\,\dot{\phi}$$
$$z = r\cos\theta\ ;\ \dot{z} = \dot{r}\cos\theta - r\sin\theta\,\dot{\theta}$$

$$T = \frac{1}{2}m\,(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

$$T = \frac{1}{2}m\,(\dot{r}^2 + r^2\,\dot{\theta}^2 + (r\sin\theta)^2\,\dot{\phi}^2)$$

+ cross terms which all sum to zero

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\,\dot{\theta}^2 + (r\sin\theta)^2\,\dot{\phi}^2) - U(r,\theta,\phi)$$



Spherical coords

27.2 Example 1: the rotating bead

A bead of mass m slides on a frictionless wire which rotates about a vertical axis at an angular velocity ω. The wire is tilted away from the vertical by an angle α. Describe the motion of the bead.

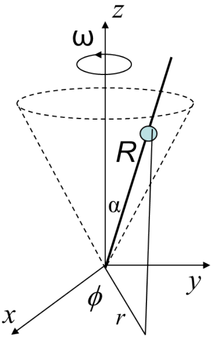
- Use spherical coordinates
- From before :

  $$T = \frac{1}{2} m \left(\dot{R}^2 + R^2 \dot{\alpha}^2 + (R \sin \alpha)^2 \dot{\phi}^2 \right)$$

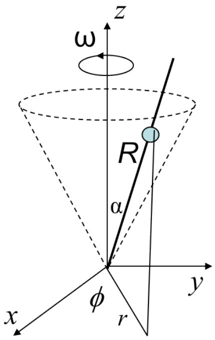
- But $\dot{\alpha} = 0$, $\dot{\phi} = \omega =$ constant
- $$T = \frac{1}{2} m \left(\dot{R}^2 + R^2 \omega^2 \sin^2 \alpha \right)$$

  $U = mgR \cos \alpha$ (Take $U = 0$ at $R = 0$)
- $L = T - U$

  $$L = \frac{1}{2} m \left(\dot{R}^2 + R^2 \omega^2 \sin^2 \alpha \right) - mgR \cos \alpha$$



The rotating bead, continued



- $L = \frac{1}{2} m \left(\dot{R}^2 + R^2 \omega^2 \sin^2 \alpha \right) - mgR \cos \alpha$
- Single generalized coordinate R
- E-L equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{R}}\right) = \frac{\partial L}{\partial R}$$

$$\frac{\partial L}{\partial R} = mR\omega^2 \sin^2 \alpha - mg \cos \alpha$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{R}}\right) = \frac{d}{dt}\left(m\dot{R}\right) = m\ddot{R}$$

  - $\ddot{R} - R\omega^2 \sin^2 \alpha = -g \cos \alpha$
  - Solution : $R = Ae^{-\lambda t} + Be^{+\lambda t} + R_0$ where $\lambda = \omega \sin \alpha$

    [$P.I. \rightarrow \ddot{R} = 0 \rightarrow R_0 = \frac{g \cos \alpha}{\omega^2 \sin^2 \alpha}$]

If $\dot{R} = 0$ at $t = 0$, $A = B$; then if $R = R_1$ at $t = 0$, $A = B = \frac{1}{2}(R_1 - R_0)$

  - If $\ddot{R} = 0 \rightarrow R = R_0 = \frac{g \cos \alpha}{\omega^2 \sin^2 \alpha} \rightarrow$ circular motion

27.3 Example 2: bead on rotating hoop

A vertical circular hoop of radius R rotates about a vertical axis at a constant angular velocity ω. A bead of mass m can slide on the hoop without friction. Describe the motion of the bead.

- Use spherical coordinates again
- From before :

  $$T = \frac{1}{2}m\left(\dot{R}^2 + R^2\dot{\theta}^2 + (R\sin\theta)^2\dot{\phi}^2\right)$$

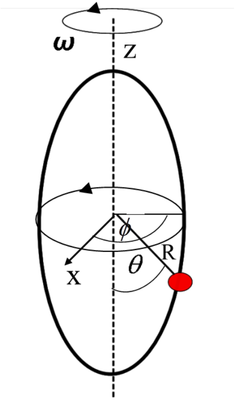
- But $\dot{R} = 0$, $\dot{\phi} = \omega =$ constant

  $$T = \frac{1}{2}m(R^2\,\dot{\theta}^2 + (R\sin\theta)^2\,\omega^2) \quad \text{(NB. } \dot{R} = 0\text{)}$$

- $U = -mgR\cos\theta$ ($U = 0$ at $\theta = 90°$)
- $L = T - U$

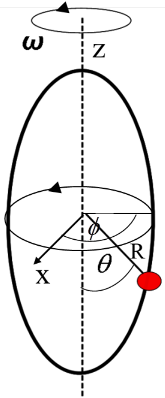
  $$L = \frac{1}{2}m(R^2\,\dot{\theta}^2 + (R\sin\theta)^2\,\omega^2) + mgR\cos\theta$$

  One single generalized coordinate : θ



Bead on rotating hoop, continued

$$L = \frac{1}{2}m(R^2\dot{\theta}^2 + R^2\sin^2\theta\,\omega^2) + mgR\cos\theta$$



- E-L equation: $\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) = \frac{\partial L}{\partial\theta}$
- $\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) = \frac{d}{dt}\left(m\,R^2\dot{\theta}\right) = m\,R^2\ddot{\theta}$

  $\frac{\partial L}{\partial\theta} = m\,R^2 sin\,\theta\cos\theta\,\omega^2 - mgR\sin\theta$

  $\rightarrow\ \ddot{\theta} = sin\,\theta\cos\theta\,\omega^2 - \frac{g}{R}\sin\theta$

  $\rightarrow\ \ddot{\theta} + \left(\omega_0^2 - \omega^2\cos\theta\right)\sin\theta = 0$

  where $\omega_0^2 = \frac{g}{R}$

  - If $\omega = 0$, $\ddot{\theta} + \omega_0^2\sin\theta = 0\ \rightarrow$ SHM, back to pendulum formula
  - If $\omega \neq 0$, for equilibrium, a necessary condition : $\ddot{\theta} = 0$

    $\rightarrow\ \theta = 0$ (stable equilibrium provided $\omega^2 R \leq g$),

    $\rightarrow\ \theta = \pi$ (unstable equilibrium)

    $\rightarrow\ \cos\theta = \frac{\omega_0^2}{\omega^2} = \frac{g}{\omega^2 R}$
    (stable equilibrium about a circle provided $\omega^2 R \geq g$)