Classical Mechanics

LECTURE 26:

THE LAGRANGE EQUATION EXAMPLES

Prof. N. Harnew

University of Oxford

HT 2017

OUTLINE : 26. THE LAGRANGE EQUATION : EXAMPLES

26.1 Conjugate momentum and cyclic coordinates

- The E-L equation is $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = \frac{\partial L}{\partial q_k}$ with $L = T - U$
- Define *conjugate (generalized) momentum* : $p_k = \frac{\partial L}{\partial \dot{q}_k}$

  Note this is not necessarily linear momentum !

  $\rightarrow$ eg. simple pendulum $L = \frac{1}{2}m\ell^2\dot{\theta}^2 + mg\ell\cos\theta$

  $\rightarrow$ $\frac{\partial L}{\partial \dot{\theta}} = m\ell^2\dot{\theta}$: which is angular momentum
- Following on, E-L equation reads $\dot{p}_k = \frac{\partial L}{\partial q_k}$
- If the Lagrangian L does not explicitly depend on q_k, the coordinate q_k is called *cyclic* or *ignorable*
- With no q_k dependence :

  $\frac{\partial L}{\partial q_k} = 0$ and $p_k = \frac{\partial L}{\partial \dot{q}_k}$ = constant

The momentum conjugate to a cyclic coordinate is a constant of motion

26.2 Example : rotating bead

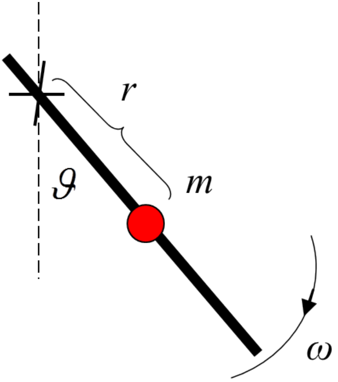
A bead slides on a wire rotating at constant angular speed ω in a horizontal plane

- Polar coordinates $\underline{\mathbf{v}} = \dot{r}\,\underline{\hat{\mathbf{r}}} + r\dot{\theta}\underline{\hat{\theta}}$
- $L = T - U$ with $U = 0$
- $L = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\omega^2$
- Single variable $q_k \to r$
- E-L $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{\partial L}{\partial r}$

  $\frac{\partial L}{\partial \dot{r}} = m\dot{r} \;\to\; \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = m\ddot{r}$

  $\frac{\partial L}{\partial r} = mr\omega^2$
- E-L $\to m\ddot{r} - mr\omega^2 = 0$

  Central force $F_{central} = m\omega^2 r$
- $r = Ae^{\omega t} + Be^{-\omega t}$



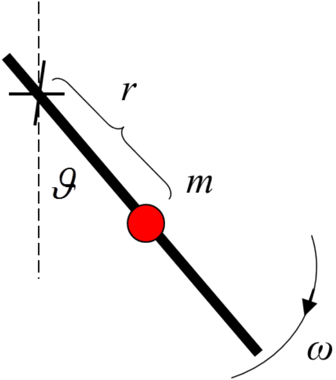
Example : rotating bead continued

What happens if the angular speed is now a free coordinate ?

- $L = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2$
- Two variables $q_k \rightarrow r,\, \theta$
- r variable: as before

  $\rightarrow m\ddot{r} - mr\dot{\theta}^2 = 0$
- θ variable: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{\partial L}{\partial \theta}$
- $\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$
- $\frac{\partial L}{\partial \theta} = 0$
- E-L : $mr^2\ddot{\theta} = \frac{d}{dt}\left(mr^2\dot{\theta}\right) = 0$

  $\rightarrow$ Conservation of angular momentum

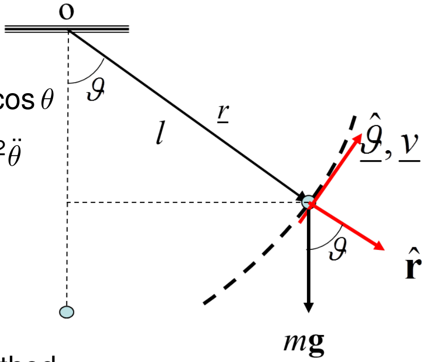


26.3 Example : simple pendulum

Evaluate simple pendulum using Euler-Lagrange equation

- Single variable $q_k \to \theta$
- $v = \ell\,\dot\theta$
- $T = \frac{1}{2} m\ell^2\dot\theta^2$
- $U = -mg\ell\cos\theta$
- $L = T - U = \frac{1}{2} m\ell^2\dot\theta^2 + mg\ell\cos\theta$
- $\frac{\partial L}{\partial\dot\theta} = m\ell^2\dot\theta \;\to\; \frac{d}{dt}\left(\frac{\partial L}{\partial\dot\theta}\right) = m\ell^2\ddot\theta$
- $\frac{\partial L}{\partial\theta} = -mg\ell\sin\theta$
- E-L $\to m\ell^2\ddot\theta + mg\ell\sin\theta = 0$

  $\to \ddot\theta + \frac{g}{\ell}\sin\theta = 0$

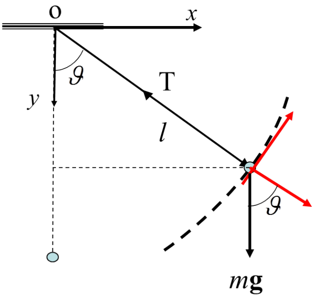


This is great, but note that the method does not get the tension in the string since ℓ is a constraint (see next slide).

26.3.1 Dealing with forces of constraint

For the simple pendulum using Euler-Lagrange equation. The method did not get the tension in the string since ℓ was constrained. If we need to find the string tension, we need to include the radial term into the Lagrangian and to include a potential function to represent the tension:

- $\ell \to r$, add $\frac{1}{2}m\dot{r}^2$, add $V(r)$
- $L = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + mgr\cos\theta - V(r)$
- $\frac{\partial L}{\partial \dot{r}} = m\dot{r} \ \to \ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = m\ddot{r}$
- $\frac{\partial L}{\partial r} = mr\dot{\theta}^2 + mg\cos\theta - \frac{\partial V(r)}{\partial r}$
- $-\frac{\partial V(r)}{\partial r} = (-T)$ with T in the $-\hat{\mathbf{r}}$ dir$^{\text{n}}$.
- E-L $\to m\ddot{r} = mr\dot{\theta}^2 + mg\cos\theta - T$
- Reintroduce $\ddot{r} = 0$ and $r = \ell$; $v = r\dot{\theta}$

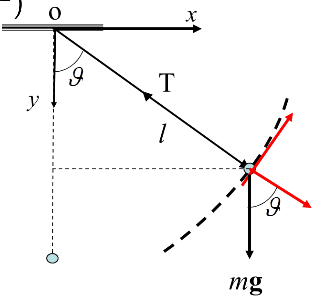


$$\underbrace{\frac{mv^2}{r}}_{\text{Centripetal force}} = \underbrace{T}_{\text{Tension}} - \underbrace{mg\cos\theta}_{\text{Weight}}$$ as expected from NII

26.3.2 The Lagrange multiplier method

An alternative method of dealing with constraints.
Back to the simple pendulum using Euler-Lagrange equation $\cdots$
Before : single variable $q_k \to \theta$. This time take TWO variables x, y but introduce a constraint into the equation. $L = T - U$

- $L' = \frac{1}{2}m(\dot{x}^2+\dot{y}^2)+mgy+\frac{1}{2}\lambda(x^2+y^2-\ell^2)$
  λ is the *Lagrange multiplier*
- $\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}_i}\right) = \frac{\partial L'}{\partial q_i}$ (including λ)
  x coord. $\to$ $m\ddot{x} = \lambda x$ (1)
  y coord. $\to$ $m\ddot{y} = mg + \lambda y$ (2)
  λ coord. $\to$ $x^2 + y^2 - \ell^2 = 0$ (3)
  (which reproduces the constraint)



Comparing with Newton II : $m\ddot{x} = -\frac{Tx}{\ell}$; $m\ddot{y} = mg - \frac{Ty}{\ell}$.
We see from the NII approach the Lagrange multiplier λ is proportional to the string tension $\lambda = -\frac{T}{\ell}$ and introduces force