Classical Mechanics

LECTURE 25:

THE LAGRANGE EQUATION DERIVED VIA THE CALCULUS OF VARIATIONS

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HT 2017

OUTLINE : 25. THE LAGRANGE EQUATION DERIVED VIA THE CALCULUS OF VARIATIONS

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25.1 The Lagrangian : simplest illustration

The Lagrangian : $L = T - U$

- In 1D : Kinetic energy $T = \frac{1}{2}m\dot{x}^2$ No explicit dependence on x
  Potential energy $U = U(x)$ No explicit dependence on $\dot{x}$
- Define the Lagrangian in 1D : $L = \frac{1}{2}m\dot{x}^2 - U(x)$
- $\frac{\partial L}{\partial \dot{x}} = m\dot{x}$ and $\frac{\partial L}{\partial x} = -\frac{\partial U}{\partial x}$ gives force F
- Differentiate wrt time : $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x} = F$
- Hence we get the Euler - Lagrange equation for x :
  $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x}$$
- Now generalize : the Lagrangian becomes a function of $2n$ variables (n is the dimension of the configuration space). Variables are the positions and velocities
  $L(q_1, \cdots, q_n, \dot{q}_1, \cdots, \dot{q}_n)$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = \frac{\partial L}{\partial q_k}$$

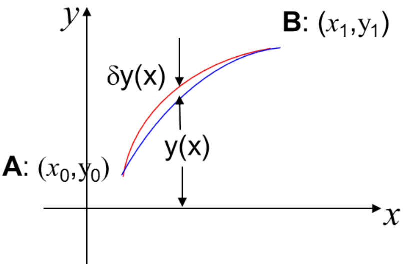
Next we expand on this concept.

25.2 The calculus of variations

- Take 2 points $A(x_0, y_0)$ and $B(x_1, y_1)$
- Curve joining them is represented by equation $y = y(x)$ such that $y(x)$ satisfies the boundary conditions :

  $\rightarrow\ y(x_0) = y_0\ ,\ y(x_1) = y_1$
- We want to find the function $y = y(x)$ subject to the above conditions which makes the closest path between the points a minimum.

  (note that this differs from what we are used to. We are not minimizing a set of variables here but a *function*).



- This is the *calculus of variations*. A branch of mathematics that deals with *functionals* as opposed to functions.

The calculus of variations, continued (1)

- We assume the unknown function f is a continuously differentiable scalar function, and the functional to be minimized depends on $y(x)$ and at most upon its first derivative $y'(x)$.
- We then wish to find the stationary values of the path between points: an integral of the form $I = \int_{x_0}^{x_1} f(y, y', x)dx$

  $\rightarrow$ $f(y, y', x)$ is a function of x, y and y' (the first derivative of y)

- Consider a small change $\delta y(x)$ in the function $y(x)$ subject to the conditions that the endpoints are unchanged :

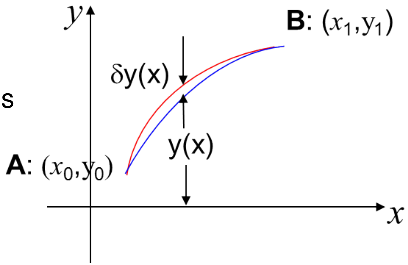
  $\rightarrow$ $\delta y(x_0) = 0$ and $\delta y(x_1) = 0$

- To first order, the variation in $f(y, y', x)$ is $\delta f = \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial y'}\delta y' + \mathcal{O}(\delta y^2, \delta y'^2)$

  where $\delta y' = \frac{d}{dx}\delta y$

- Thus the variation in the integral I is

$$\delta I = \int_{x_0}^{x_1} \left[\frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial y'}\frac{d}{dx}\delta y\right] dx$$



The calculus of variations, continued (2)

- $\delta I = \int_{x_0}^{x_1} \left[\frac{\partial f}{\partial y} \delta y + \frac{\partial f}{\partial y'} \frac{d}{dx} \delta y \right] \, dx$
- Integrate the second term by parts

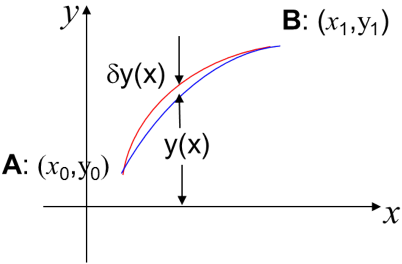
$$\text{2nd term} = \left[\frac{\partial f}{\partial y'} \delta y \right]_{x_0}^{x_1} - \int_{x_0}^{x_1} \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) \delta y \, dx$$

The $\left[\frac{\partial f}{\partial y'} \delta y \right]_{x_0}^{x_1}$ term $= 0$ due to the conditions on the end points

- Hence

$$\delta I = \int_{x_0}^{x_1} \left[\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) \right] \delta y \, dx$$

- For I to be stationary, $\delta I = 0$ for any small arbitrary variation $\delta y(x)$

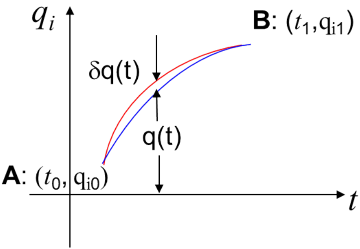


  - This is only possible if the integrand vanishes identically
  - Hence we get out the Euler-Lagrange Equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) = 0$$

The calculus of variations, continued (3)

- So far we have used x as the independent variable with a functional f which is a function of $(y(x), y', x)$
- Throughout we could have used *other* variables, in particular time t and generalized coordinates $q_1, \cdots, q_n$ and derivatives $\dot{q}_1, \cdots, \dot{q}_n$. The principles would have been the same.
- The integral $I = \int_{t_0}^{t_1} f\left[q_1(t), \cdots, q_n(t), \dot{q}_1(t), \cdots, \dot{q}_n(t)\right] dt$ must be stationary wrt variations in any one & all of the variables $q_1(t), \cdots, q_n(t)$ subject to the conditions $\delta q_i(t_0) = \delta q_i(t_1) = 0$



  - We get the n Euler-Lagrange equations for $i = 1, \cdots, n$

$$\frac{\partial f}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{q}_i}\right) = 0$$

The E-L equations give the conditions for the closest path between points

25.3 A sanity check

The shortest distance between 2 points.

- Consider 2 neighboring points on the curve $y(x)$ subject to boundary conditions $y(x_0) = y_0$, $y(x_1) = y_1$
- Distance between the points $d\ell = \sqrt{dx^2 + dy^2}$
- $d\ell = \sqrt{1 + y'^2}\, dx \qquad f \equiv \sqrt{1 + y'^2}$
- The Euler-Lagrange Equation $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$
- $\frac{\partial f}{\partial y} = 0$, $\frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{1+y'^2}}$, $\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$
- Hence $\frac{y'}{\sqrt{1+y'^2}}$ = constant, hence y' is constant $\rightarrow \quad y = mx + c$
- We have proved that the shortest distance between 2 points is a straight line !

25.4 Fermat's Principle & Snell's Law

Fermat : *The actual path that a light ray propagating between one point to another will take is the one that makes the time travelled between the two points stationary.*

Question: at which point $(x, 0)$ will the ray hit the interface between the two media to propagate from A to B?

- Time taken from A to B :

$$t(x) = \frac{1}{v_1}\left[(x-x_1)^2 + y_1^2\right]^{\frac{1}{2}} + \frac{1}{v_2}\left[(x_2-x)^2 + y_2^2\right]^{\frac{1}{2}}$$

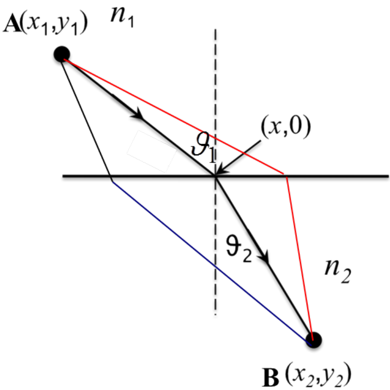
- The Euler-Lagrange Equation

$$\frac{\partial t}{\partial x} - \frac{d}{dy}\left(\frac{\partial t}{\partial x'}\right) = 0 \quad \text{(where the second term = 0)}$$

- $\frac{\partial t}{\partial x} = 0 = \frac{1}{v_1}\frac{x-x_1}{\left[(x-x_1)^2+y_1^2\right]^{\frac{1}{2}}} - \frac{1}{v_2}\frac{x_2-x}{\left[(x_2-x)^2+y_2^2\right]^{\frac{1}{2}}}$
- $\frac{\sin\theta_1}{v_1} - \frac{\sin\theta_2}{v_2} = 0$

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

Snell's Law



25.5 Hamilton's principle (Principle of Stationary Action)

- Consider for example a particle of mass m at point (x_A, y_A) moving under the influence of a force in the $x - y$ plane. We want to find the path that the particle will follow to reach a point (x_B, y_B).
- Hamilton's principle: *the path that the particle will take from A to B is the one that makes the following functional stationary* :

  $I = \int_A^B L\,(q_1(t), \cdots, q_n(t), \dot{q}_1(t), \cdots, \dot{q}_n(t))\,dt$

  where L is the Lagrangian, I is called the *action integral*
- Hence the action integral I is stationary under arbitrary variations $q_1(t), q_2(t) \cdots$ which vanish at the limits of integration ie. A and B.