Classical Mechanics

LECTURE 24:
LAGRANGE MECHANICS

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OUTLINE : 24. LAGRANGE MECHANICS

23.5 Example 3 : an aircraft landing

The landing wheel of an aircraft may be approximated as a uniform circular disk of diameter 1 m and mass 200 kg. The total mass of the aircraft including that of the 10 wheels is 100, 000 kg. When landing the touch-down speed is 50 ms^{-1}. Assume that the wheels support 50% of the total weight of the aircraft.

Determine the time duration of wheel-slip if the coefficient of friction between the wheels and the ground is 0.5*, assuming that the speed of the plane is not changed significantly.*

An aircraft landing continued

- Torque $|\underline{\tau}| = |\underline{\mathbf{r}} \times \underline{\mathbf{F}}| = a\mu M' g$ about O, where $M' = \frac{M}{20}$ (ie. 10 wheels, supporting 50% of mass)
- Angular momentum $J = \mathtt{I}\omega$ where $\mathtt{I} = \frac{1}{2}ma^2$ (MoI of solid disk where m is mass of a wheel)
- $\tau = \frac{dJ}{dt} = \mathtt{I}\frac{d\omega}{dt}$ where $u = a\omega$

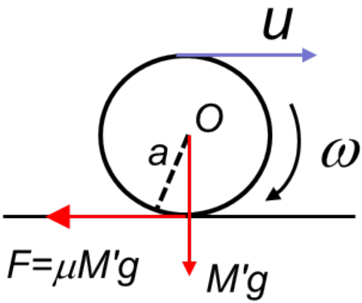
  u is the speed of the wheel rim

  $\rightarrow \tau = \frac{\mathtt{I}}{a}\frac{du}{dt} = a\mu M' g$
- Integrate : $\int_0^{t_f} \frac{a^2 \mu M' g}{\mathtt{I}} dt = \int_0^{v_0} du$

  v_0 is the speed of the aeroplane

  $$\rightarrow \boxed{t_f = \frac{v_0 \, \mathtt{I}}{a^2 \mu M' g} = \frac{v_0 \, m}{2\mu M' g}}$$

  - Putting in numbers: $t_f = \frac{50 \times 200}{2 \times 0.5 \times \frac{1 \times 10^5}{20} \times 9.8}$ $\boxed{\sim 0.2\,\mathrm{s}}$

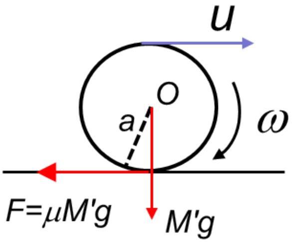


An aircraft landing continued

Confirm the assumption that the speed of the plane is not changed by calculating the speed at the end of wheel-slip in the absence of other braking processes.

- Energy expended in getting the wheels up to speed: $E_{wheels} = \frac{1}{2}\mathrm{I}\omega^2 \times 10$ (ie. 10 wheels) $= \frac{5}{2}mv_0^2$
  [Remember $v_0 = a\omega$, $\mathrm{I} = \frac{1}{2}ma^2$]
- Total energy of the aeroplane
  $E = \frac{1}{2}Mv^2$
  $\rightarrow$ Energy loss: $\delta E = Mv\,\delta v = \frac{5}{2}mv_0^2$
  $\rightarrow \delta v = \frac{\frac{5}{2}mv_0^2}{Mv_0} \rightarrow \frac{\delta v}{v_0} = \frac{\frac{5}{2}m}{M}$
- Putting in numbers : $\frac{\delta v}{v_0} = \frac{\frac{5}{2}\times 200}{1\times 10^5}$

$\rightarrow$ a 0.5% effect

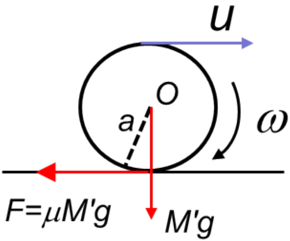


An aircraft landing continued

Calculate the work done during wheel slip.

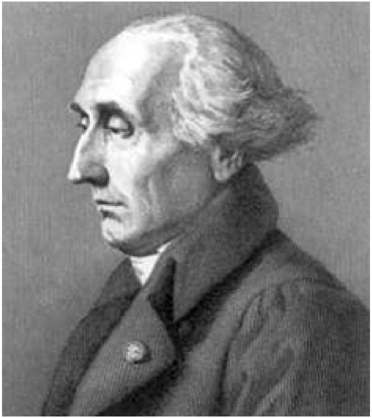
- Work done : $W = \int \tau \, d\theta = \tau \, \theta_f$ (τ is constant, θ_f is the total turning angle)
- From before : $u = a\frac{d\theta}{dt} = \frac{2\mu M' g}{m} t$
- Integrate : $\int_0^{\theta_f} a \, d\theta = \int_0^{t_f} \frac{2\mu M' g}{m} t \, dt$

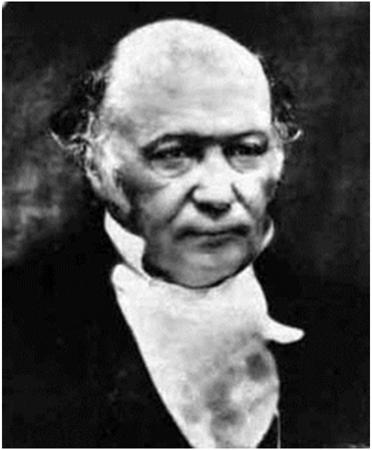
  $\rightarrow a\,\theta_f = \frac{\mu M' g}{m} t_f^2$ where $t_f = \frac{v_0 \, m}{2\mu M' g}$



- Putting it all together:
  - $W = \tau \, \theta_f = \underbrace{\frac{1}{a}(a\mu M' g)}_{\tau} \underbrace{\left(\frac{M' g \mu}{m}\right)\left(\frac{v_0 \, m}{2\mu M' g}\right)^2}_{\theta_f} = \frac{1}{2}\mathrm{I}\left(\frac{v_0}{a}\right)^2$
  - Hence $W = E_{wheel} = \frac{1}{2}\mathrm{I}\,\omega^2$ as expected !

Lagrange and Hamilton





- Joseph-Louis Lagrange (1736-1810)
- Sir William Rowan Hamilton (1805-1865)

24.1 Lagrangian mechanics : Introduction

- Lagrangian Mechanics: a very effective way to find the equations of motion for complicated dynamical systems using a scalar treatment

  $\rightarrow$ Newton's laws are vector relations. The Lagrangian is a single scalar function of the system variables

- Avoid the concept of force

  $\rightarrow$ For complicated situations, it may be hard to identify all the forces, especially if there are constraints

- The Lagrangian treatment provides a framework for relating conservation laws to symmetry
- The ideas may be extended to most areas of fundamental physics (special and general relativity, electromagnetism, quantum mechanics, quantum field theory ....)

24.2 Introductory example : the energy method for the E of M

- For conservative forces in 1D motion :

  Energy of system : $E = \frac{1}{2}m\dot{x}^2 + U(x)$ [Note $\frac{dE}{dt} = 0$]
- Differentiate wrt time: $m\dot{x}\ddot{x} + \frac{\partial U}{\partial x}\,\dot{x} = 0$

  $\rightarrow\ m\,\ddot{x} = -\frac{\partial U}{\partial x} = F$

  $\rightarrow$ This is the E of M for a conservative force
- Take a simple 1d spring undergoing SHM :

  $E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 =$ constant

  $\frac{dE}{dt} = 0 \ \rightarrow\ m\dot{x}\ddot{x} + kx\dot{x} = 0$

  $\rightarrow\ m\ddot{x} + kx = 0$
- Hence we derived the E of M without using NII directly.

24.3 Becoming familiar with the jargon

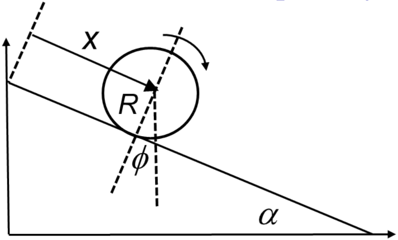
24.3.1 Generalised coordinates

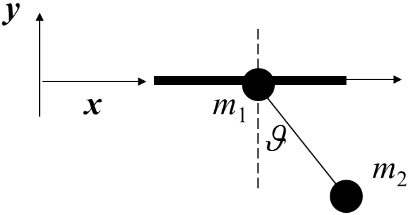
A set of parameters $q_k(t)$ that specifies the system configuration. q_k may be a geometrical parameter, x, y, z, a set of angles $\cdots$ etc

24.3.2 Degrees of Freedom

The number of degrees of freedom is the number of independent coordinates that is sufficient to describe the configuration of the system uniquely.

Examples of degrees of freedom





- Ball rolling down an incline

$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}\mathrm{I}\dot{\phi}^2 - mgx\sin\alpha$

- But $\dot{x} = R\dot{\phi} \rightarrow x = R\phi$
- The problem is reduced to a 1-coordinate variable

  $q_1 \equiv x$ and $\dot{q}_1 \equiv \dot{x}$
- System has only 1 degree of freedom : x

- Pendulum whose pivot can move freely in x direction
- Pivot coordinates : $(x, 0)$
- Pendulum coordinates : $(x + \ell\sin\theta, -\ell\cos\theta)$

$$E = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\left(\frac{d}{dt}(x + \ell\sin\theta)\right)^2 + \\ + \frac{1}{2}m_2\left(\frac{d}{dt}(-\ell\cos\theta)\right)^2 - mg\ell\cos\theta$$

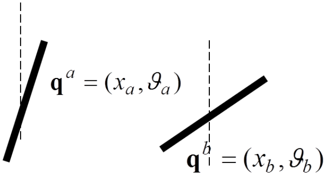
- This system has 2 degrees of freedom : x and θ

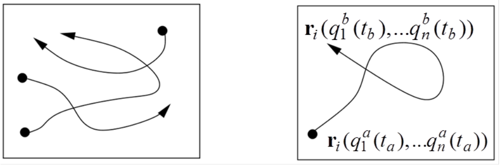
24.3.3 Constraints

- A system has *constraints* if its components are not permitted to move freely in 3-D. For example :
  - $\rightarrow$ A particle on a table is restricted to move in 2-D
  - $\rightarrow$ A mass on a simple pendulum is restricted to oscillate at an angle θ at a fixed length ℓ from a pivot
- The constraints are *Holonomic* if :
  - $\rightarrow$ The constraints are time independent
  - $\rightarrow$ The system can be described by relations between general coordinate variables and time
  - $\rightarrow$ The number of general coordinates is reduced to the number of degrees of freedom

24.3.4 Configuration Space

- The configuration space of a mechanical system is an n-dimensional space whose points determine the spatial position of the system in time. This space is parametrized by generalized coordinates, $\mathbf{q} = (q_1 \cdots, q_n)$
- Example 1. A point in space determines where the system is; the coordinates are simply standard Euclidean coordinates: $(x, y, z) = (q_1, q_2, q_3)$
- Example 2. A rod location x, angle θ - as it moves in 2D space is passes through points (x, θ) in the configuration space





Real space   Configuration space