Classical Mechanics
LECTURE 23:
MoI THEOREMS AND EXAMPLES

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OUTLINE : 23. MoI THEOREMS AND EXAMPLES

23.1 Parallel axis theorem

$\mathtt{I}_{CM}$ is the moment of inertia of body mass M about an axis passing through its centre of mass. $\mathtt{I}$ is the moment of inertia of the body about a parallel axis a distance d from the first.

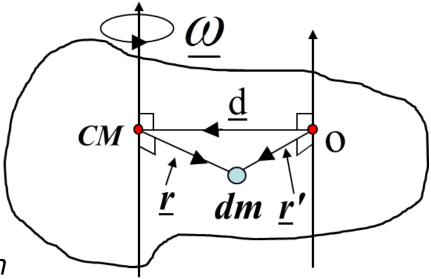
- About the axis through the CM

  $\mathtt{I}_{CM} = \int r^2 dm$
- $\underline{\mathbf{r}}' = \underline{\mathbf{d}} + \underline{\mathbf{r}}$
- $r'^2 = d^2 + 2\,\underline{\mathbf{d}} \cdot \underline{\mathbf{r}} + r^2$
- About the parallel axis :
  $\mathtt{I} = \int r'^2 dm$

$$= \int d^2 dm + 2\,\underline{\mathbf{d}} \cdot \underbrace{\int \underline{\mathbf{r}}\, dm}_{=0} + \int r^2 dm$$

(definition of CM)

- Hence $\mathtt{I} = \mathtt{I}_{CM} + Md^2$



23.1.1 Example : compound pendulum

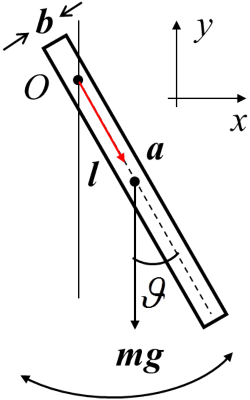
Rectangular rod length a width b mass m swinging about axis O, distance ℓ from the CM, in plane of paper

- $\mathtt{I}_{CM} = m\,(\frac{a^2+b^2}{12})$
- Parallel axis theorem :

  $\mathtt{I} = \mathtt{I}_{CM} + m\ell^2$
- Torque about O $= \underline{\mathbf{r}} \times \underline{\mathbf{F}}$:

  $\underline{\tau} = -m\,g\,\ell \sin\theta\,\hat{\underline{\mathbf{k}}}$
- $\underline{\mathbf{J}} = \mathtt{I}\,\underline{\omega} = \mathtt{I}\,\dot{\theta}\,\hat{\underline{\mathbf{k}}}$
- Differentiate : $\underline{\tau} = \frac{d\mathbf{J}}{dt} = \mathtt{I}\,\ddot{\theta}\,\hat{\underline{\mathbf{k}}}$
- Equate $\mathtt{I}\,\ddot{\theta} = -m\,g\,\ell \sin\theta$
- Small angle approximation

  $\ddot{\theta} + \frac{m\,g\,\ell}{\mathtt{I}}\theta = 0$

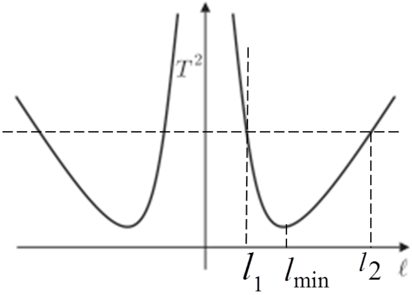


Compound pendulum continued

- $\ddot{\theta} + \frac{m\,g\,\ell}{\mathtt{I}}\theta = 0$

  where $\mathtt{I} = m\,(\frac{a^2+b^2}{12}) + m\,\ell^2$
- SHM with period $\quad T = 2\pi\sqrt{\frac{\mathtt{I}}{mg\ell}}$

$$\rightarrow \quad T = 2\pi\sqrt{\frac{a^2+b^2+12\ell^2}{12g\ell}}$$



- Measurement of g :
- Plot T^2 vs ℓ
- Find values of ℓ_1 and ℓ_2 that give the same value of $T \rightarrow T_1 = T_2$

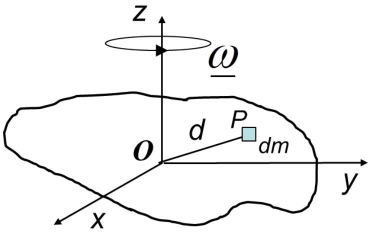
Can show : $\frac{T_1^2}{(\ell_1+\ell_2)} = \frac{4\pi^2}{g}$

23.2 Perpendicular axis theorem

Consider a rigid object that *lies entirely within a plane*. The perpendicular axis theorem links $\mathtt{I}_z$ (MoI about an axis perpendicular to the plane) with $\mathtt{I}_x$, $\mathtt{I}_y$ (MoI about two perpendicular axes lying within the plane).

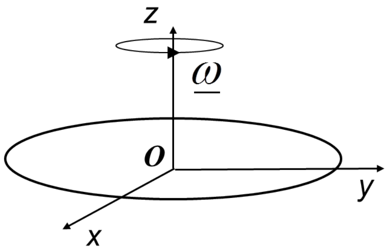
- Consider perpendicular axes x , y , z (which meet at origin O) ; the body lies in the xy plane
- $\mathtt{I}_z = \int d^2\, dm$
  $= \int \left(x^2 + y^2\right)\, dm$
  $= \int x^2\, dm + \int y^2\, dm$
  $\rightarrow \boxed{\mathtt{I}_z = \mathtt{I}_x + \mathtt{I}_y}$

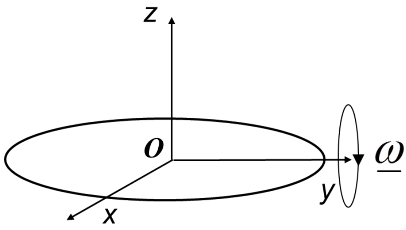
  This is the *perpendicular axis theorem*.



23.2.1 Perpendicular axis theorem : example

- Consider a thin circular disk lying in a plane
- Perpendicular axis theorem : $\mathtt{I}_z = \mathtt{I}_x + \mathtt{I}_y$
- $\mathtt{I}_z = \frac{1}{2} M R^2$ (R = radius of disk)
- Hence $\mathtt{I}_x = \mathtt{I}_y = \frac{1}{4} M R^2$ (due to symmetry)





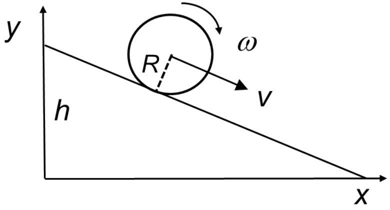
23.3 Example 1 : solid ball rolling down slope

[Energy of ball] = [Rotational KE in CM] + [KE of CM] + [PE]

$$E = \frac{1}{2}\mathtt{I}\omega^2 + \frac{1}{2}Mv^2 + Mgy$$

- Ball falls a distance h from rest $\rightarrow$ at $y = 0$:

$$M\,g\,h = \frac{1}{2}\mathtt{I}\omega^2 + \frac{1}{2}Mv^2$$
$$= \frac{1}{2}\mathtt{I}\left(\frac{v}{R}\right)^2 + \frac{1}{2}Mv^2$$



- Solid sphere: $\mathtt{I} = \frac{2}{5}M\,R^2$
- $Mgh = \frac{1}{2}Mv^2\left(\frac{2}{5} + 1\right)$

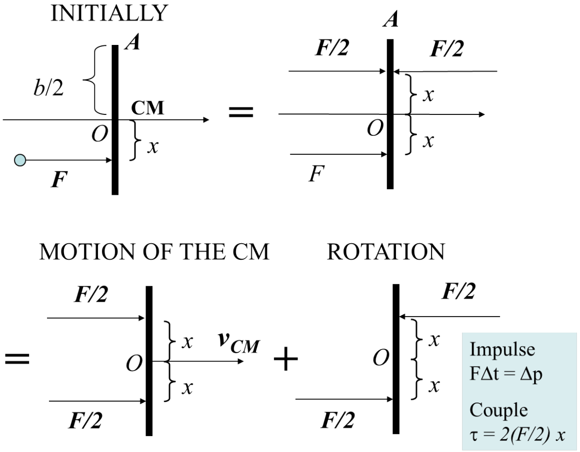
$$\rightarrow \quad v = \sqrt{\frac{10}{7}gh}$$

Compare with a solid cylinder $\quad \mathtt{I} = \frac{1}{2}M\,R^2 \quad \rightarrow \quad v = \sqrt{\frac{4}{3}gh}$

The ball gets to the bottom faster !

23.4 Example 2 : where to hit a ball with a cricket bat

We want the bat handle (Point A, $\frac{b}{2}$ from the CM) to remain stationary after the ball has hit. When the ball hits, there is rotation of the bat about the CM, plus motion of the CM of the bat. The velocity to the right (v_{CM}) must equal the velocity from the left $\frac{\omega b}{2}$ due to rotation.

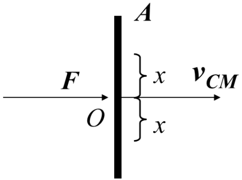


Where to hit a ball with a cricket bat continued

- Ball hits at point x from centre
- Force to the CM : $F = ma \ \rightarrow a = \frac{F}{m}$
- Moment of inertia wrt CM :
  $\mathtt{I}_{CM} = \frac{1}{12} mb^2$
- Torque (couple) about O $= \mathtt{I}_{CM}\,\ddot{\theta}$
- Hence $\mathtt{I}_{CM}\,\ddot{\theta} = \frac{mb^2}{12}\ddot{\theta} = x \times \frac{F}{2} \times 2$
  $\rightarrow \ddot{\theta} = \frac{12Fx}{mb^2}$
- Require acceleration at A to be zero.
- Acceleration at A due to rotation = $\frac{b}{2}\ddot{\theta}$
- Equate accelerations :
  $$\frac{F}{m} = \frac{6Fx}{mb} \rightarrow \ x = \frac{b}{6}$$

Need to hit the bat $\frac{2}{3}$ from the top

MOTION OF THE CM



ROTATION