Classical Mechanics

LECTURE 22:

ROTATIONAL DYNAMICS

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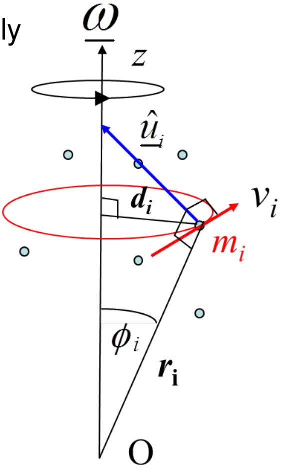
OUTLINE : 22. ROTATIONAL DYNAMICS

22.1 Moment of inertia tensor

- Consider bodies rotating around a common axis $\underline{\omega}$, no axis of symmetry, $\underline{\mathbf{J}}$ not necessarily parallel to $\underline{\omega}$, origin O lies on the $\underline{\omega}$ ($\underline{\mathbf{z}}$) axis.
- Definition of angular velocity : $\underline{\dot{\mathbf{r}}} = \underline{\omega} \times \underline{\mathbf{r}}$
- $\underline{\mathbf{J}} = \sum_i \underline{\mathbf{r}}_i \times \underline{\mathbf{p}}_i = \sum_i m_i \underline{\mathbf{r}}_i \times \underline{\dot{\mathbf{r}}}_i$
  $= \sum_i m_i \underline{\mathbf{r}}_i \times (\underline{\omega} \times \underline{\mathbf{r}}_i)$
- Use the vector identity
  $\underline{\mathbf{a}} \times (\underline{\mathbf{b}} \times \underline{\mathbf{c}}) = (\underline{\mathbf{a}}.\underline{\mathbf{c}})\,\underline{\mathbf{b}} - (\underline{\mathbf{a}}.\underline{\mathbf{b}})\,\underline{\mathbf{c}}$

$$\underline{\mathbf{J}} = \sum_i m_i r_i^2 \underline{\omega} - \sum_i m_i (\underline{\mathbf{r}}_i \cdot \underline{\omega})\,\underline{\mathbf{r}}_i$$

[Note for circular motion in a plane where $\underline{\mathbf{r}}_i$ is $\perp$ to $\underline{\omega}$, $\sum_i m_i (\underline{\mathbf{r}}_i \cdot \underline{\omega}) = 0$, and hence in this case $\underline{\mathbf{J}}$ is $\parallel$ to $\underline{\omega}$.]



Moment of inertia tensor continued

- From before : $\underline{\mathbf{J}} = \sum_i m_i r_i^2 \underline{\omega} - \sum_i m_i (\underline{\mathbf{r}}_i \cdot \underline{\omega}) \underline{\mathbf{r}}_i$
- Can express in terms of components

$$(J_x, J_y, J_z) = \sum_i m_i (x_i^2 + y_i^2 + z_i^2)(\omega_x, \omega_y, \omega_z) - \sum_i m_i (x_i \omega_x + y_i \omega_y + z_i \omega_z)(x_i, y_i, z_i)$$

- Construct a matrix equation :

$$\begin{pmatrix} J_x \\ J_y \\ J_z \end{pmatrix} = \begin{pmatrix} \sum_i (y_i^2 + z_i^2) m_i & -\sum_i (x_i y_i) m_i & -\sum_i (x_i z_i) m_i \\ -\sum_i (x_i y_i) m_i & \sum_i (x_i^2 + z_i^2) m_i & -\sum_i (y_i z_i) m_i \\ -\sum_i (x_i z_i) m_i & -\sum_i (y_i z_i) m_i & \sum_i (x_i^2 + y_i^2) m_i \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

If mass is continuous (rigid body) :

$$\begin{pmatrix} J_x \\ J_y \\ J_z \end{pmatrix} = \begin{pmatrix} \int (y^2 + z^2)\, dm & -\int xy\, dm & -\int xz\, dm \\ -\int xy\, dm & \int (x^2 + z^2)\, dm & -\int yz\, dm \\ -\int xz\, dm & -\int yz\, dm & \int (x^2 + y^2)\, dm \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

- Hence we can write : $\underline{\mathbf{J}} = \widetilde{\mathtt{I}} \underline{\omega}$
- $\widetilde{\mathtt{I}}$ is the *Moment of Inertia tensor* of the system

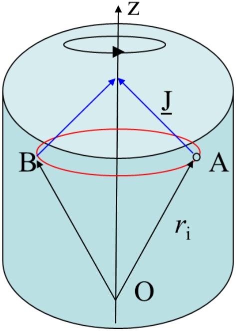
22.1.1 Rotation about a principal axis

- In general $\underline{\mathbf{J}} = \tilde{\mathrm{I}}\,\underline{\omega}$, where $\tilde{\mathrm{I}}$ is the *Moment of Inertia Tensor*

$$\begin{pmatrix} J_x \\ J_y \\ J_z \end{pmatrix} = \begin{pmatrix} \mathrm{I}_{xx} & \mathrm{I}_{xy} & \mathrm{I}_{xz} \\ \mathrm{I}_{yx} & \mathrm{I}_{yy} & \mathrm{I}_{yz} \\ \mathrm{I}_{zx} & \mathrm{I}_{zy} & \mathrm{I}_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

- Whenever possible, one aligns the axes of the coordinate system in such a way that the mass of the body evenly distributes around the axes: we choose *axes of symmetry*.

$$\begin{pmatrix} J_x \\ J_y \\ J_z \end{pmatrix} = \begin{pmatrix} \mathrm{I}_{xx} & 0 & 0 \\ 0 & \mathrm{I}_{yy} & 0 \\ 0 & 0 & \mathrm{I}_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$



The diagonal terms are called the *principal axes* of the moment of inertia.

- Whenever we rotate about an axis of symmetry, for every point A there is a point B which cancels it, and

$$\underline{\mathbf{J}} \rightarrow J_z\,\hat{\underline{\mathbf{z}}} = \mathrm{I}_{zz}\,\omega\,\hat{\underline{\mathbf{z}}}$$

and where $\underline{\mathbf{J}}$ is parallel to $\underline{\omega}$ along the $\underline{\mathbf{z}}$ axis

22.2 Moment of inertia & energy of rotation

Particles rotating in circular motion about a common axis of rotation with angular velocity $\underline{\omega}$ (where $\underline{\mathbf{v}}_i = \underline{\omega} \times \underline{\mathbf{r}}_i$).

- Kinetic energy of mass m_i :

  $$T_i = \frac{1}{2} m_i v_i^2$$

- Total KE = $\frac{1}{2} \sum_i \left(m_i v_i^2 \right)$
- $\underline{\mathbf{v}}_i = \underline{\omega} \times \underline{\mathbf{r}}_i$

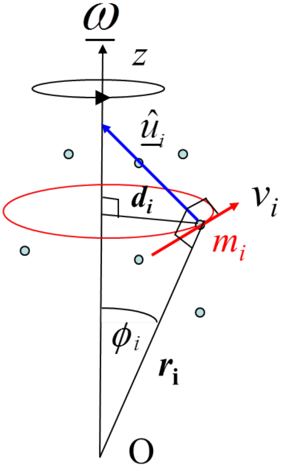
  $$v_i = \omega\, r_i \sin \phi_i$$

  $$\sin \phi_i = \frac{d_i}{r_i}$$

- $T_{rot} = \frac{1}{2} \left(\sum_i^N m_i\, d_i^2 \right) \omega^2$

  $$T_{rot} = \frac{1}{2} \mathtt{I}_z\, \omega^2$$

  where $\mathtt{I}_z$ is calculated about the axis of rotation



22.3.1 MoI of a thin rectangular plate

(a) About the x axis

- $\mathtt{I}_x = \int y^2\, dm$

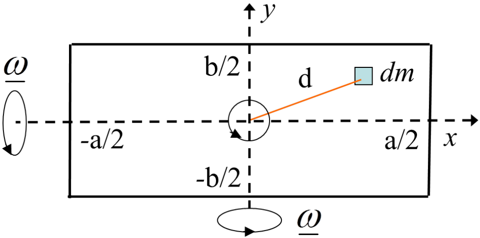
$dm = \rho\, dx\, dy$; $\rho = \frac{M}{ab}$

- $\mathtt{I}_x = \int y^2\, \rho\, dx\, dy$

$$= \left[\rho\,\frac{y^3}{3}\right]_{-\frac{b}{2}}^{+\frac{b}{2}} [x]_{-\frac{a}{2}}^{+\frac{a}{2}}$$

$$= \rho\, a \left[\frac{b^3}{24} + \frac{b^3}{24}\right] = \rho\, a\, b \left[\frac{b^2}{12}\right]$$

- Hence $\mathtt{I}_x = \frac{M\, b^2}{12}$



(b) About the y axis

- $\mathtt{I}_y = \frac{M\, a^2}{12}$

(c) About the z axis

- $\mathtt{I}_z = \int \rho\, d^2\, dx\, dy$

$$= \int \rho\, (x^2 + y^2)\, dx\, dy$$

$$= \left[\rho\,\frac{y^3}{3}\right]_{-\frac{b}{2}}^{+\frac{b}{2}} + \left[\frac{x^3}{3}\right]_{-\frac{a}{2}}^{+\frac{a}{2}}$$

- Hence $\mathtt{I}_z = M\,(\frac{a^2+b^2}{12})$

22.3.2 MoI of a thin disk perpendicular to plane of disk

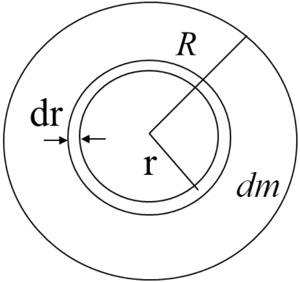
- $I_z = \int r^2 \, dm$

$dm = \rho(2\pi r \, dr)$; $\rho = \frac{M}{\pi R^2}$

- $I_z = 2\pi\rho \int r^3 dr$

$= \left[2\pi\rho \frac{r^4}{4}\right]_0^R = (\pi \frac{R^4}{2}) \frac{M}{\pi R^2}$

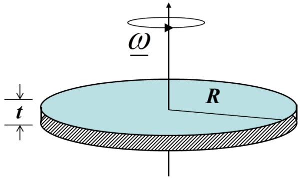
- Hence $I_z = \frac{1}{2} M R^2$



- For a cylinder thickness t :
- Use cylindrical coordinates

$dm = \rho(2\pi r \, dr \, dt)$; $\rho = \frac{M}{\pi R^2 t}$

- $I_z = 2\pi\rho \int \int r^3 dr \, dt$
- $I_z = \frac{1}{2} M R^2$, the same.



22.3.3 MoI of a solid sphere

- $d\mathrm{I} = \frac{1}{2}x^2\,dm$ (for a disk)

$dm = \rho(\pi x^2\,dz)$; $\rho = \frac{M}{\frac{4}{3}\pi R^3}$

$\rightarrow \quad d\mathrm{I} = \frac{\pi\rho}{2}x^4\,dz$

- $\mathrm{I} = \int d\mathrm{I} = \frac{1}{2}\pi\rho\int_{-R}^{+R} x^4 dz$
- But $x^2 = R^2 - z^2$

$\mathrm{I} = 2\times\frac{1}{2}\pi\rho\int_0^{+R}(R^4 - 2R^2z^2 + z^4)dz$

$= \pi\rho\left[R^4z - \frac{2R^2z^3}{3} + \frac{1}{5}z^5\right]_0^R$

$= \pi\rho\,\frac{8}{15}R^5 = M(\pi\,\frac{8}{15}R^5)/(\frac{4}{3}\pi R^3)$

- Hence $\mathrm{I} = \frac{2}{5}M\,R^2$