Classical Mechanics

LECTURE 21:

SYSTEMS OF PARTICLES AND MOMENT OF INERTIA

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OUTLINE : 21. SYSTEMS OF PARTICLES AND MOMENT OF INERTIA

21.1 NII for system of particles - translation motion

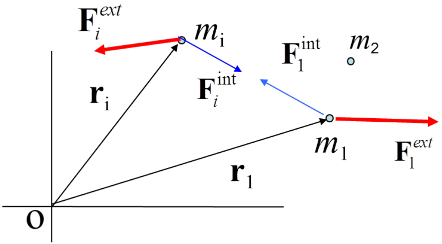
Reminder from MT lectures:

- Force on particle i: $\quad m_i \frac{d^2}{dt^2}(\underline{\mathbf{r}}_\mathbf{i}) = \underline{\mathbf{F}}_\mathbf{i}{}^{ext} + \underline{\mathbf{F}}_\mathbf{i}{}^{int}$
- $$\underbrace{\sum_i^N m_i \frac{d^2}{dt^2}(\underline{\mathbf{r}}_\mathbf{i})}_{\text{all masses}} = \underbrace{\sum_i^N \underline{\mathbf{F}}_\mathbf{i}{}^{ext}}_{\text{external forces}} + \underbrace{\sum_i^N \underline{\mathbf{F}}_\mathbf{i}{}^{int}}_{\text{internal forces = zero}} = \sum_i^N \underline{\mathbf{F}}_\mathbf{i}{}^{ext}$$

- $\underline{\mathbf{r}}_{CM} = \sum_i^N \frac{m_i \underline{\mathbf{r}}_\mathbf{i}}{M}$

  where $M = \sum_i^N m_i$
- $\underline{\mathbf{v}}_{CM} = \dot{\underline{\mathbf{r}}}_{CM} = \sum_i^N \frac{m_i \dot{\underline{\mathbf{r}}}_i}{M}$

  $\rightarrow \underline{\mathbf{P}}_{CM} = \sum_i^N m_i \dot{\underline{\mathbf{r}}}_\mathbf{i} = M \underline{\mathbf{v}}_{CM}$
- $\underline{\mathbf{a}}_{CM} = \ddot{\underline{\mathbf{r}}}_{CM} = \sum_i^N \frac{m_i \ddot{\underline{\mathbf{r}}}_i}{M} = \frac{\underline{\mathbf{F}}_\mathbf{i}{}^{ext}}{M}$



21.1.1 Kinetic energy and the CM

- Lab kinetic energy : $T = \frac{1}{2}\sum_i^N m_i\,\underline{\mathbf{v}}_i^2 \quad ; \quad \underline{\mathbf{v}}_i = \underline{\mathbf{v}}_i' + \underline{\mathbf{v}}_{CM}$

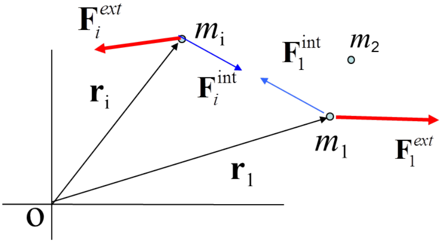
  where $\underline{\mathbf{v}}_i'$ is velocity of particle i in the CM

- $T = \frac{1}{2}\sum_i m_i\,\underline{\mathbf{v}}_i'^2 + \frac{1}{2}\sum_i m_i\,\underline{\mathbf{v}}_{CM}^2 + \sum_i m_i\,\underline{\mathbf{v}}_i' \cdot \underline{\mathbf{v}}_{CM}$

- But $\sum_i m_i\underline{\mathbf{v}}_i' \cdot \underline{\mathbf{v}}_{CM} = \underbrace{\frac{\sum_i m_i\underline{\mathbf{v}}_i'}{M}}_{=0} \cdot M\,\underline{\mathbf{v}}_{CM}$

- $\boxed{T = T' + \frac{1}{2}M\underline{\mathbf{v}}_{CM}^2}$

Same expression as was derived in MT

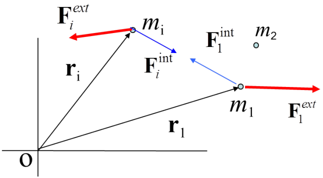


21.2 NII for system of particles - rotational motion

- Angular momentum of particle i about O : $\underline{\mathbf{J}}_i = \underline{\mathbf{r}}_i \times \underline{\mathbf{p}}_i$
- Torque of i about O: $\underline{\tau}_i = \frac{d\underline{\mathbf{J}}_i}{dt} = \underline{\mathbf{r}}_i \times \underline{\dot{\mathbf{p}}}_i + \underbrace{\underline{\dot{\mathbf{r}}}_i \times \underline{\mathbf{p}}_i}_{=0} = \underline{\mathbf{r}}_i \times \underline{\mathbf{F}}_i$

- Total ang. mom. of system

$$\underline{\mathbf{J}} = \sum_i^N \underline{\mathbf{J}}_i = \sum_i^N \underline{\mathbf{r}}_i \times \underline{\mathbf{p}}_i$$

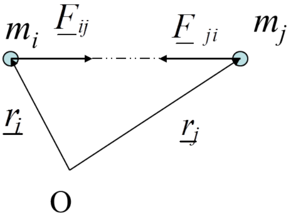


- Internal forces:

$$\sum_{pair}^{int} \underline{\tau}_{(i,j)} = \underline{\mathbf{r}}_i \times \underline{\mathbf{F}}_{ij}^{int} + \underline{\mathbf{r}}_j \times \underline{\mathbf{F}}_{ji}^{int}$$

$$= (\underline{\mathbf{r}}_i - \underline{\mathbf{r}}_j) \times \underline{\mathbf{F}}_{ij}^{int}$$

$= 0$ since $(\underline{\mathbf{r}}_i - \underline{\mathbf{r}}_j)$ parallel to $\underline{\mathbf{F}}_{ij}^{int}$



Hence total torque

$$\underline{\tau} = \sum_i^N (\underline{\mathbf{r}}_i \times \underline{\mathbf{F}}_i^{ext}) = \frac{d\underline{\mathbf{J}}}{dt}$$

If external torque $= 0$, $\underline{\mathbf{J}}$ is const.

21.2.1 Angular momentum and the CM

- Lab to CM : $\underline{\mathbf{r}}_i = \underline{\mathbf{r}}'_i + \underline{\mathbf{r}}_{CM}$; $\underline{\mathbf{v}}_i = \underline{\mathbf{v}}'_i + \underline{\mathbf{v}}_{CM}$

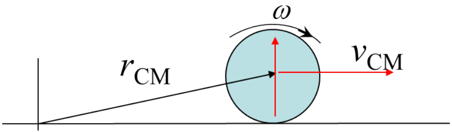
  where $\underline{\mathbf{r}}'_i$, $\underline{\mathbf{v}}'_i$ are position & velocity of particle i wrt the CM
- Total ang. mom. of system

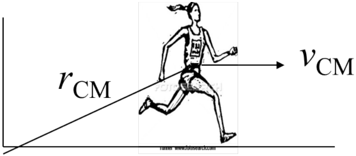
$$\underline{\mathbf{J}} = \sum_i^N \underline{\mathbf{J}}_i = \sum_i^N m_i\,(\underline{\mathbf{r}}'_i + \underline{\mathbf{r}}_{CM}) \times (\underline{\mathbf{v}}'_i + \underline{\mathbf{v}}_{CM})$$

$$= \sum_i m_i\,(\underline{\mathbf{r}}'_i \times \underline{\mathbf{v}}'_i) + \sum_i m_i\,(\underline{\mathbf{r}}'_i \times \underline{\mathbf{v}}_{CM}) + \sum_i m_i\,(\underline{\mathbf{r}}_{CM} \times \underline{\mathbf{v}}'_i) + \sum_i m_i\,(\underline{\mathbf{r}}_{CM} \times \underline{\mathbf{v}}_{CM})$$

But $\sum_i m_i\,(\underline{\mathbf{r}}'_i \times \underline{\mathbf{v}}_{CM}) = \underbrace{[\sum_i m_i\,\underline{\mathbf{r}}'_i]}_{= 0 \text{ in CM}} \times \underline{\mathbf{v}}_{CM}$; $\sum_i m_i\,(\underline{\mathbf{r}}_{CM} \times \underline{\mathbf{v}}'_i) = \underline{\mathbf{r}}_{CM} \times \underbrace{[\sum_i m_i\,\underline{\mathbf{v}}'_i]}_{= 0 \text{ in CM}}$

- Hence $\boxed{\underline{\mathbf{J}} = \underbrace{\underline{\mathbf{J}}'}_{\text{J wrt CM}} + \underbrace{\underline{\mathbf{r}}_{CM} \times M\,\underline{\mathbf{v}}_{CM}}_{\text{J of CM translation}}}$





What we have learned so far

- Newton's laws relate to rotating systems in the same way that the laws relate to transitional motion.
- For any system of particles, the rate of change of internal angular momentum about an origin is equal to the total torque of the external forces about the origin.
- The total angular momentum about an origin is the sum of the total angular momentum about the CM plus the angular momentum of the translation of the CM.

21.3 Introduction to Moment of Inertia

- Take the simplest example of 2 particles rotating in circular motion about a common *axis of rotation* with angular velocity $\underline{\omega} = \omega\,\underline{\hat{\mathbf{z}}}$
- Definition of $\underline{\omega}$ for circular motion : $\boxed{\underline{\dot{\mathbf{r}}} = \underline{\omega} \times \underline{\mathbf{r}}}$

- Total angular momentum of the system of particles about O

  $\underline{\mathbf{J}} = \underline{\mathbf{r}}_1 \times (m_1 \underline{\mathbf{v}}_1) + \underline{\mathbf{r}}_2 \times (m_2 \underline{\mathbf{v}}_2)$

- $\underline{\mathbf{v}}_1 = \underline{\omega} \times \underline{\mathbf{r}}_1$

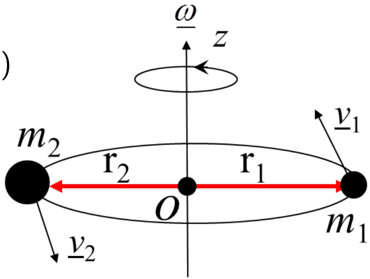
  $\underline{\mathbf{v}}_2 = \underline{\omega} \times \underline{\mathbf{r}}_2$

- Since $\underline{\mathbf{r}}_i \perp \underline{\mathbf{v}}_i$

  $\underline{\mathbf{J}} = (m_1\, r_1^2 + m_2\, r_2^2)\,\underline{\omega}$

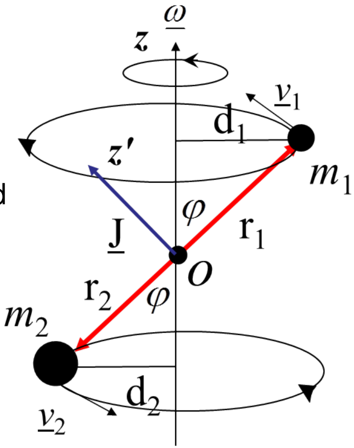
- $\underline{\mathbf{J}} = \mathtt{I}\underline{\omega}$ ($\underline{\mathbf{J}}$ is parallel to $\underline{\omega}$)
  - Moment of Inertia $\mathtt{I} = m_1\, r_1^2 + m_2\, r_2^2$

    Or more generally $\mathtt{I} = \sum_i [m_i\, r_i^2]$



21.3.1 Extend the example : J not parallel to ω

Now consider the same system but with rotation tilted wrt rotation axis by an angle ϕ. Again $\underline{\omega} = \omega\,\hat{\underline{\mathbf{z}}}$



- Total angular momentum of the particles about O :

  $\underline{\mathbf{J}} = \underline{\mathbf{r}}_1 \times (m_1 \underline{\mathbf{v}}_1) + \underline{\mathbf{r}}_2 \times (m_2 \underline{\mathbf{v}}_2)$
- NB. $\underline{\mathbf{J}}$ now points along the $\underline{\mathbf{z}}'$ axis
- $\underline{\mathbf{J}}$ vector is $\perp$ to line of m_1 and m_2 and defines the *principal axis* of the MoI mass distribution (see later)
- Since $v_1 = d_1\,\omega \;\; ; \;\; r_1 = \frac{d_1}{\sin\phi}$

  $v_2 = d_2\,\omega \;\; ; \;\; r_2 = \frac{d_2}{\sin\phi}$
- Then $|\underline{\mathbf{J}}| = \frac{(m_1\, d_1^2 + m_2\, d_2^2)\,\omega}{\sin\phi}$

  $\rightarrow |\underline{\mathbf{J}}| \sin\phi\, \hat{\underline{\omega}} = \mathtt{I}_z\, \underline{\omega}$. Hence $J_z = \mathtt{I}_z \omega$ *where* $\mathtt{I}_z$ *is calculated about the* $\hat{\underline{\mathbf{z}}}$ *(i.e.* $\hat{\underline{\omega}}$*) axis.*

21.3.2 Moment of inertia : mass not distributed in a plane

Now take a system of particles rotating in circular motion about a common axis of rotation, all with angular velocity $\underline{\omega}$ (where $\underline{\mathbf{v}}_i = \underline{\omega} \times \underline{\mathbf{r}}_i$).

- Total angular momentum of the system of particles about O (which is on the axis of rotation)

  $\underline{\mathbf{J}} = \sum_i^N \underline{\mathbf{r}}_i \times (m_i \underline{\mathbf{v}}_i) = \sum_i^N m_i r_i v_i \, \underline{\hat{\mathbf{u}}}_\mathbf{i}$

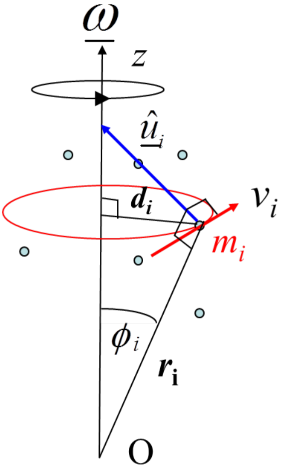
  (not necessarily parallel to $\underline{\omega}$ axis)

- As before, resolve the angular momentum about the axis of rotation

  $J_z \, \underline{\hat{\mathbf{z}}} = \sum_i^N m_i r_i v_i \sin\phi_i \, \underline{\hat{\omega}}$

- $\underline{\mathbf{v}}_i = \underline{\omega} \times \underline{\mathbf{r}}_i$; $v_i = \omega r_i \sin\phi_i$

  Also $\sin\phi_i = \frac{d_i}{r_i}$; $v_i = \omega \, d_i$



$J_z \, \underline{\hat{\omega}} = \left(\sum_i^N m_i \, d_i^2\right) \underline{\omega} \;\rightarrow\; J_z = \mathtt{I}_z \, \omega$ ($\mathtt{I}_z$: MoI about rot^n axis)

21.3.3 Generalize for rigid bodies

A rigid body may be considered as a collection of infinitesimal point particles whose relative distance does not change during motion.

- $\sum_i m_i \to \int dm,\ where\ dm = \rho dV$ and ρ is the volume density
- $\mathtt{I}_z = \left(\sum_i^N m_i\, d_i^2\right) \ \to\ \int_V d^2\, \rho\, dV$
- This integral gives the moment of inertia about axis of rotation (z axis)