Classical Mechanics

LECTURE 20:

OPEN ORBITS

Prof. N. Harnew
University of Oxford

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OUTLINE : 20. OPEN ORBITS

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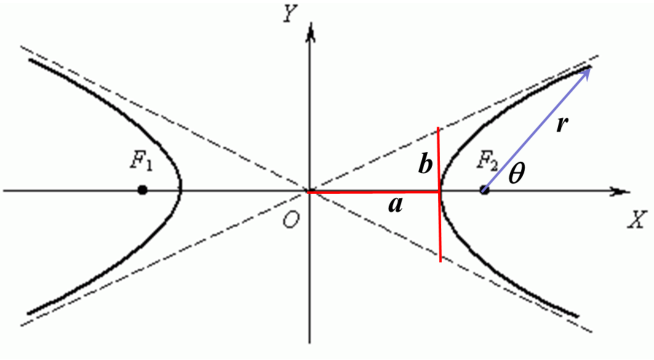
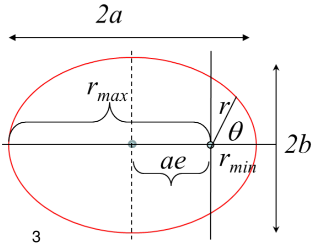
20.1 The hyperbolic orbit



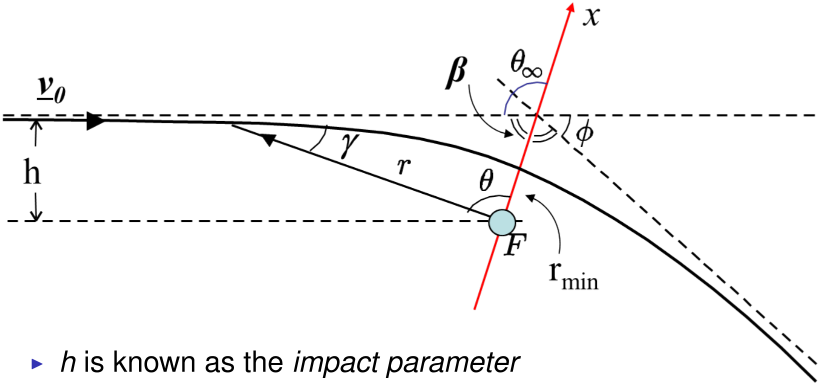


- Orbit equation : $r(\theta) = \frac{r_0}{1+e\cos\theta}$
- Ellipse : $e < 1 \quad \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$
- Hyperbola :
  $e > 1 \quad \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1$

When $x \& y \to \infty, \quad \frac{y}{b} = \frac{x}{a} \to y = (\frac{b}{a})x$

20.2 Hyperbolic orbit : the distance of closest approach

For example a comet deviated by the gravitational attraction of a planet. Velocity $\underline{\mathbf{v}} = \underline{\mathbf{v}}_0$ when $\underline{\mathbf{r}} \to \infty$.



- h is known as the *impact parameter*
- Angular momentum $\underline{\mathbf{J}} = m\,\underline{\mathbf{r}} \times \underline{\mathbf{v}}$

  $\to |\underline{\mathbf{J}}| = mvr\sin\gamma = mv_0 h$ (as $r \to \infty$)

  $\to$ Total energy $E = \frac{1}{2} m v_0^2$ (again as $r \to \infty$)

Distance of closest approach continued

- $E = \frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} - \frac{\alpha}{r}$, where $\alpha = GmM$
- At distance of closest approach $r = r_{min} \rightarrow \dot{r} = 0$
- $E = \frac{J^2}{2mr_{min}^2} - \frac{\alpha}{r_{min}}$

  $\rightarrow r_{min}^2 + \frac{\alpha}{E} r_{min} - \frac{J^2}{2mE} = 0$

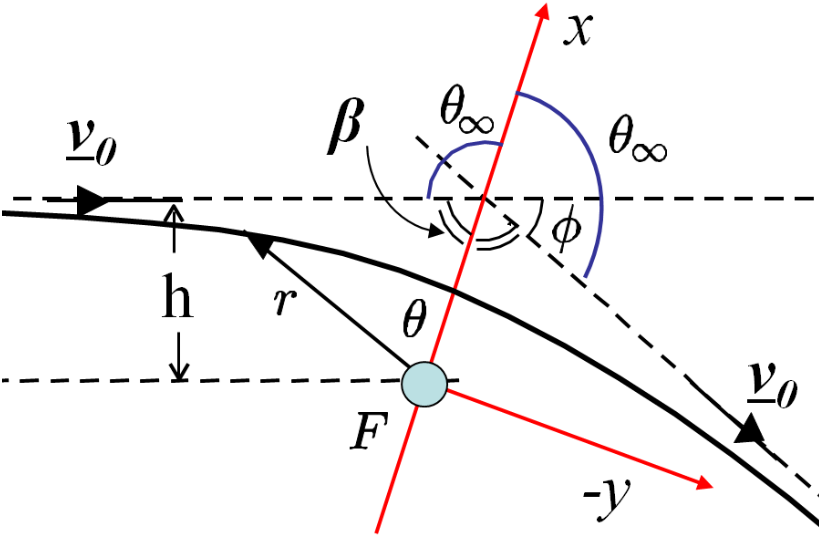
- Same form of solution as for the ellipse :
- $r_{min} = -\left(\frac{\alpha}{2E}\right)[1 - \underbrace{(1 + \frac{2EJ^2}{m\alpha^2})^{\frac{1}{2}}}_{e}]$ $\quad (J^2 = (mv_0 h)^2 \; ; \; E = \frac{1}{2} m v_0^2)$

  $\rightarrow \quad r_{min} = \underbrace{-\frac{\alpha}{2E}(1 - e)}_{= a(e - 1)}$

- Velocity v' at distance of closest approach: line to trajectory is a right angle.

  $\rightarrow \; J = mv' r_{min} = mv_0 h \quad \rightarrow \quad v' = \frac{v_0 h}{r_{min}}$

20.3 Hyperbolic orbit: the angle of deflection, ϕ



20.3.1 Method 1 : using impulse

- Directly from the diagram : $\Delta v_x = 2v_0 \cos\theta_\infty$ (1)
- By symmetry, integrated change in $v_y = 0$: $\Delta v_y = 0$
- Change in Δp_x : $m\Delta v_x = \int_{-\infty}^{+\infty} F_x dt = \int_{-\infty}^{+\infty} F_x \underbrace{\left(\frac{mr^2\dot{\theta}}{J}\right)}_{=1} dt$

  $\rightarrow \quad m\Delta v_x = \left(\frac{m}{J}\right) \int_{-\theta_\infty}^{+\theta_\infty} F_x r^2 d\theta$
- But $\underline{\mathbf{F}} = -\left(\frac{\alpha}{r^2}\right)\underline{\hat{\mathbf{r}}} \quad \rightarrow \quad F_x = -\frac{\alpha}{r^2}\cos\theta$

  $\rightarrow \quad m\Delta v_x = -2\left(\frac{m\alpha}{J}\right) \int_0^{\theta_\infty} \cos\theta d\theta$

  $\rightarrow \quad \Delta v_x = -\left(\frac{2\alpha}{J}\right)\sin\theta_\infty$ (2)
- From (1) & (2) $\rightarrow -\left(\frac{2\alpha}{J}\right)\sin\theta_\infty = 2v_0 \cos\theta_\infty$
- $\tan\theta_\infty = -\frac{Jv_0}{\alpha} \quad \rightarrow \quad \theta_\infty + \beta = \pi \; ; \; \phi + 2\beta = \pi$
- $\theta_\infty = \frac{\phi}{2} + \frac{\pi}{2} \; ; \; \tan\theta_\infty = \tan\left(\frac{\phi}{2} + \frac{\pi}{2}\right) = -\cot\frac{\phi}{2}$

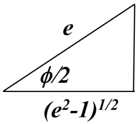
$$\cot\frac{\phi}{2} = \frac{J\,v_0}{\alpha} = \frac{mhv_0^2}{\alpha} = \frac{hv_0^2}{GM}$$

20.3.2 Method 2 : using hyperbola orbit parameters

- Orbit equation : $r(\theta) = \frac{r_0}{1+e\cos\theta}$
- $r \to \infty \quad , \quad \cos\theta_\infty = -\frac{1}{e}$
- $\theta_\infty + \beta = \pi\ ; \quad \phi + 2\beta = \pi \ \to\ \frac{\phi}{2} = \theta_\infty - \frac{\pi}{2} \ \to\ \sin\frac{\phi}{2} = \frac{1}{e}$

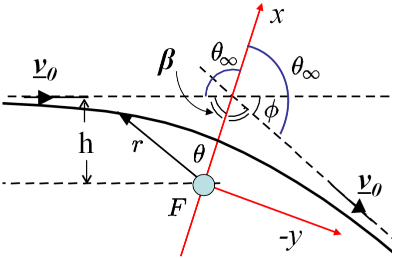
- From before :

$$r_{min} = -\left(\frac{\alpha}{2E}\right)\left[1 - \underbrace{\left(1 + \frac{2EJ^2}{m\alpha^2}\right)^{\frac{1}{2}}}_{e}\right]$$



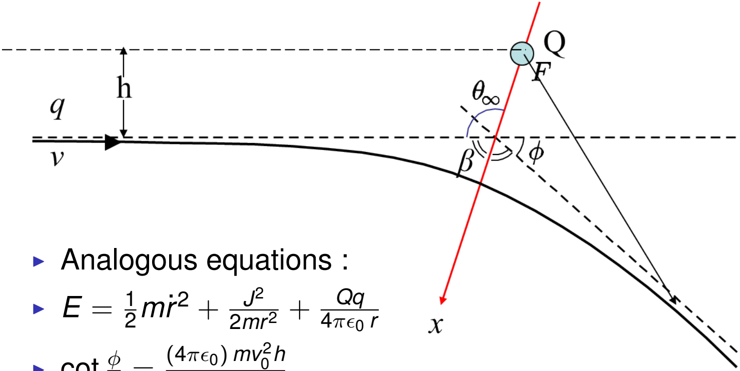
- $\cot\frac{\phi}{2} = [e^2 - 1]^{\frac{1}{2}} = [\frac{2EJ^2}{m\alpha^2}]^{\frac{1}{2}}$
- $E = \frac{1}{2}mv_0^2\ ;\ J = mv_0 h$
- $\cot\frac{\phi}{2} = \frac{mv_0^2 h}{\alpha} = \frac{v_0^2 h}{GM}$

as before



20.4 Hyperbolic orbit : Rutherford scattering

A particle of charge $+q$ deviated by the Coulomb repulsion of a nucleus, charge $+Q$. Analogous hyperbolic motion as previously: $\alpha = GMm \;\rightarrow\; \alpha = -\frac{Qq}{4\pi\epsilon_0}$. As before, velocity $\underline{\mathbf{v}} = \underline{\mathbf{v_0}}$ when $\underline{\mathbf{r}} \rightarrow \infty$. Here we take the other branch of the hyperbola.



- Analogous equations :
- $E = \frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} + \frac{Qq}{4\pi\epsilon_0\, r}$
- $\cot\frac{\phi}{2} = \frac{(4\pi\epsilon_0)\, m v_0^2 h}{qQ}$
- $r_{min} = \left(\frac{qQ}{2E(4\pi\epsilon_0)}\right)[1 + (1 + \frac{2EJ^2}{m\alpha^2})^{\frac{1}{2}}]$