OUTLINE: 2. MORE ON VECTORS

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2.1.1 The scalar (dot) product

Scalar (or dot) product definition:
\[ \mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| \cdot |\mathbf{b}| \cos \theta \equiv ab \cos \theta \]

(write shorthand \(|\mathbf{a}| = a\).

- Scalar product is the magnitude of \(\mathbf{a}\) multiplied by the projection of \(\mathbf{b}\) onto \(\mathbf{a}\).
- Obviously if \(\mathbf{a}\) is perpendicular to \(\mathbf{b}\) then \(\mathbf{a} \cdot \mathbf{b} = 0\).
- Also \(\mathbf{a} \cdot \mathbf{a} = |a|^2\) (since \(\theta = 0^\circ\))
  Hence \(a = \sqrt{\mathbf{a} \cdot \mathbf{a}}\)
Properties of scalar product

(i) \( i \cdot i = j \cdot j = k \cdot k = 1 \) and \( i \cdot j = j \cdot k = k \cdot i = 0 \)

(ii) This leads to \( a \cdot b = (a_x i + a_y j + a_z k) \cdot (b_x i + b_y j + b_z k) \)
    \[ = a_x b_x + a_y b_y + a_z b_z \]

(iii) \( a \cdot b = b \cdot a \): commutative
    \( a \cdot (b + c) = a \cdot b + a \cdot c \): distributive

(iv) Parentheses are important
    Note \( (u \cdot v) w \neq u (v \cdot w) \) because one is a vector along \( \hat{w} \),
    the other is along \( \hat{u} \).
2.1.2 The vector (cross) product

Vector (or cross) product of two vectors, definition:

\[ \mathbf{a} \times \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \sin \theta \mathbf{\hat{n}} \]

where \( \mathbf{\hat{n}} \) is a unit vector in a direction perpendicular to both \( \mathbf{a} \) and \( \mathbf{b} \).

To get direction of \( \mathbf{a} \times \mathbf{b} \) use right hand rule:

- i) Make a set of directions with your right hand → thumb & first index finger, and with middle finger positioned perpendicular to plane of both
- ii) Point your thumb along the first vector \( \mathbf{a} \)
- iii) Point your 1st index finger along \( \mathbf{b} \), making the smallest possible angle to \( \mathbf{a} \)
- iv) The direction of the middle finger gives the direction of \( \mathbf{a} \times \mathbf{b} \).
Properties of vector product

- \( \mathbf{i} \times \mathbf{j} = \mathbf{k} \); \( \mathbf{j} \times \mathbf{k} = \mathbf{i} \); \( \mathbf{k} \times \mathbf{i} = \mathbf{j} \); \( \mathbf{i} \times \mathbf{i} = 0 \) etc.

- \( (\mathbf{a} + \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \times \mathbf{c}) + (\mathbf{b} \times \mathbf{c}) \) : distributive

- \( \mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a} \) : NON-commutative

- \( (\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \neq \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \) : NON-associative

- If \( m \) is a scalar,
  \[ m(\mathbf{a} \times \mathbf{b}) = (ma) \times \mathbf{b} = \mathbf{a} \times (mb) = (\mathbf{a} \times \mathbf{b})m \]

- \( \mathbf{a} \times \mathbf{b} = 0 \) if vectors are parallel \( (0^\circ) \)
  i.e \( \mathbf{a} \times \mathbf{a} = 0 \)
Vector product in components

Cross product written out in components:

\[ \mathbf{a} \times \mathbf{b} = (a_x, a_y, a_z) \times (b_x, b_y, b_z) \]
\[ = (a_x \mathbf{i} + a_y \mathbf{j} + a_z \mathbf{k}) \times (b_x \mathbf{i} + b_y \mathbf{j} + b_z \mathbf{k}) \]
\[ = (a_y b_z - a_z b_y) \mathbf{i} - (a_x b_z - a_z b_x) \mathbf{j} + (a_x b_y - a_y b_x) \mathbf{k} \]

Since \( \mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0 \) and \( \mathbf{i} \times \mathbf{j} = \mathbf{k} \) etc.

This is much easier when we write in determinant form:

\[ \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} \] (1)
2.1.3 Examples of scalar & vector products in mechanics

▶ a) Scalar product

*Work done* on a body by a force through distance $\text{d}r$ from position 1 to 2

$$W_{12} = \int_1^2 F \cdot \text{d}r$$

Only the component of force parallel to the line of displacement does work.

▶ b) Vector product

A *torque* about O due to a force $\mathbf{F}$ acting at B:

$$\tau = \mathbf{r} \times \mathbf{F}$$

Torque is a vector with direction perpendicular to both $\mathbf{r}$ and $\mathbf{F}$, magnitude of $|\mathbf{r}| |\mathbf{F}| \sin \theta$. 
2.2 Differentiation of vectors

Notation: a dot above the function indicates derivative wrt time. A “dash” indicates derivative wrt a spatial coordinate.

\[ \dot{y} \equiv \frac{dy}{dt} \quad y' \equiv \frac{dy}{dx} \]

\[ \dot{\mathbf{a}} = \lim_{\Delta t \to 0} \frac{\mathbf{a}(t + \Delta t) - \mathbf{a}(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{a}}{\Delta t} \]

\[ \mathbf{a}(t) = a_x(t) \mathbf{i} + a_y(t) \mathbf{j} + a_z(t) \mathbf{k} \]

\[ \dot{\mathbf{a}} = \lim_{\Delta t \to 0} \left( \frac{a_x(t + \Delta t) - a_x(t)}{\Delta t} \mathbf{i} + \ldots \right) \]

Hence \[ \dot{\mathbf{a}} = \dot{a}_x \mathbf{i} + \dot{a}_y \mathbf{j} + \dot{a}_z \mathbf{k} \]
2.2.1 Vector velocity

- $\Delta r = r_2 - r_1$
- $v = \lim_{\Delta t \to 0} \frac{\Delta r}{\Delta t}$
- Velocity at any point is tangent to the path at that point
- $\mathbf{v} = \frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}}$

In one dimension:
Abandon vector notation and simply write $v = \frac{dx}{dt} = \dot{x}$,
($+v$ in $+x$ direction, $-v$ in $-x$ direction).
Example - 1D motion

A body has velocity \( v_0 = -15 \text{ ms}^{-1} \) at position \( x_0 = 20 \text{ m} \) and has a time-dependent acceleration \( a(t) = 6t - 4 \text{ [ms}^{-2}\text{]} \). Find the value of \( x \) for which the body instantaneously comes to rest.

\[
\begin{align*}
\text{a}(t) &= 6t - 4 \text{ [ms}^{-2}\text{]} ; \quad x_0 = 20 \text{ m} ; \quad v_0 = -15 \text{ ms}^{-1} \\
\dot{v} &= 6t - 4 \quad \Rightarrow \quad v = \int a(t)dt = 3t^2 - 4t + c \\
\text{At } t = 0, v &= -15 \text{ ms}^{-1} \quad \Rightarrow \quad c = -15 \text{ ms}^{-1} \\
v &= 3t^2 - 4t - 15 \\
v &= 0 \text{ for } 3t^2 - 4t - 15 = 3(t - 3)(t + \frac{5}{3}) = 0 \\
&\quad \Rightarrow t = 3 \text{s (also } -\frac{5}{3} \text{s)} \\
x &= \int v(t)dt = t^3 - 2t^2 - 15t + c' \quad \Rightarrow \quad x = 20 \text{ m at } t = 0, c' = 20 \text{ m} \\
x(t) &= 27 - 18 - 45 + 20 = -16 \text{ m}
\end{align*}
\]
2.2.2 Vector acceleration

\[ \Delta v = v_2 - v_1 \]

\[ a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \dot{v} = \ddot{x} \]

In one dimension:
Abandon vector notation and simply write \( a = \frac{dv}{dt} = \dot{v} = \ddot{x}, \)
(+\( a \) in +\( x \) direction, −\( a \) in −\( x \) direction).
**Example: constant acceleration - projectile motion in 2D**

\[ a = \frac{dv}{dt} = \text{constant} \]

\[ \mathbf{r} = 0 \text{ at } t = 0 \]

\[ \int_{v_0}^{v} dv = \int_{0}^{t} a \, dt \]

\[ \rightarrow v = v_0 + at \quad \rightarrow \mathbf{v} = \frac{d\mathbf{r}}{dt} \]

\[ \int_{0}^{t} d\mathbf{r} = \int_{0}^{t} (v_0 + at) \, dt \]

\[ \rightarrow \mathbf{r} = v_0 t + \frac{1}{2} at^2 \]

**Under gravity:** \[ \mathbf{a} = -g \mathbf{\hat{y}} \quad \rightarrow \quad a_x = 0; \quad a_y = -g \]

\[ v_x = v_0 \cos \theta \]

\[ v_y = v_0 \sin \theta - gt \]

\[ x = (v_0 \cos \theta) t \]

\[ y = (v_0 \sin \theta) t - \frac{1}{2} gt^2 \]

**Trajectory:** \[ y = (\tan \theta) x - \frac{g}{2v_0^2} (\sec^2 \theta) x^2 \]
The monkey and the hunter