Classical Mechanics

LECTURE 2:
MORE ON VECTORS

Prof. N. Harnew
University of Oxford

MT 2016

OUTLINE : 2. MORE ON VECTORS

2.1 Scalar and vector products

2.1.1 The scalar (dot) product

2.1.2 The vector (cross) product

2.1.3 Examples of scalar & vector products in mechanics

2.2 Differentiation of vectors

2.2.1 Vector velocity

2.2.2 Vector acceleration

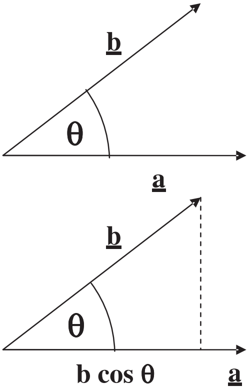
2.1.1 The scalar (dot) product

Scalar (or dot) product definition:

$\underline{\mathbf{a}}.\underline{\mathbf{b}} = |\underline{\mathbf{a}}|.|\underline{\mathbf{b}}| \cos\theta \equiv ab \cos\theta$

(write shorthand $|\underline{\mathbf{a}}| = $ a).

- Scalar product is the magnitude of $\underline{\mathbf{a}}$ multiplied by the projection of $\underline{\mathbf{b}}$ onto $\underline{\mathbf{a}}$.
- Obviously if $\underline{\mathbf{a}}$ is perpendicular to $\underline{\mathbf{b}}$ then $\underline{\mathbf{a}}.\underline{\mathbf{b}} = 0$
- Also $\underline{\mathbf{a}}.\underline{\mathbf{a}} = |a|^2$ (since θ =0°)
  Hence a = $\sqrt{(\underline{\mathbf{a}}.\underline{\mathbf{a}})}$



Properties of scalar product

(i) $\underline{\mathbf{i}}.\underline{\mathbf{i}} = \underline{\mathbf{j}}.\underline{\mathbf{j}} = \underline{\mathbf{k}}.\underline{\mathbf{k}} = 1$ and $\underline{\mathbf{i}}.\underline{\mathbf{j}} = \underline{\mathbf{j}}.\underline{\mathbf{k}} = \underline{\mathbf{k}}.\underline{\mathbf{i}} = 0$

(ii) This leads to $\underline{\mathbf{a}}.\underline{\mathbf{b}} = (a_x\underline{\mathbf{i}} + a_y\underline{\mathbf{j}} + a_z\underline{\mathbf{k}}).(b_x\underline{\mathbf{i}} + b_y\underline{\mathbf{j}} + b_z\underline{\mathbf{k}})$
= $a_x b_x + a_y b_y + a_z b_z$

iii) $\underline{\mathbf{a}}.\underline{\mathbf{b}} = \underline{\mathbf{b}}.\underline{\mathbf{a}}$: commutative
$\underline{\mathbf{a}}.(\underline{\mathbf{b}} + \underline{\mathbf{c}}) = \underline{\mathbf{a}}.\underline{\mathbf{b}} + \underline{\mathbf{a}}.\underline{\mathbf{c}}$: distributive

(iv) Parentheses are important
Note $(\underline{\mathbf{u}}.\underline{\mathbf{v}})\ \underline{\mathbf{w}} \neq \underline{\mathbf{u}}\ (\underline{\mathbf{v}}.\underline{\mathbf{w}})$ because one is a vector along $\hat{\underline{\mathbf{w}}}$, the other is along $\hat{\underline{\mathbf{u}}}$.

2.1.2 The vector (cross) product

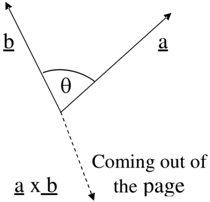
Vector (or cross) product of two vectors, definition:

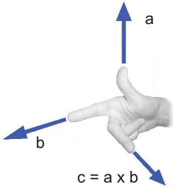
$$\underline{\mathbf{a}} \times \underline{\mathbf{b}} = |\underline{\mathbf{a}}||\underline{\mathbf{b}}| \sin\theta \; \underline{\hat{\mathbf{n}}}$$

where $\underline{\hat{\mathbf{n}}}$ is a *unit vector* in a direction *perpendicular* to both $\underline{\mathbf{a}}$ and $\underline{\mathbf{b}}$.

To get direction of $\underline{\mathbf{a}} \times \underline{\mathbf{b}}$ use right hand rule:

- i) Make a set of directions with your *right hand*$\rightarrow$ thumb & first index finger, and with middle finger positioned perpendicular to plane of both
- ii) Point your thumb along the first vector $\underline{\mathbf{a}}$
- iii) Point your 1st index finger along $\underline{\mathbf{b}}$, making the smallest possible angle to $\underline{\mathbf{a}}$
- iv) The direction of the middle finger gives the direction of $\underline{\mathbf{a}} \times \underline{\mathbf{b}}$.





Properties of vector product

- $\underline{\mathbf{i}} \times \underline{\mathbf{j}} = \underline{\mathbf{k}}$; $\underline{\mathbf{j}} \times \underline{\mathbf{k}} = \underline{\mathbf{i}}$; $\underline{\mathbf{k}} \times \underline{\mathbf{i}} = \underline{\mathbf{j}}$; $\underline{\mathbf{i}} \times \underline{\mathbf{i}} = 0$ etc.
- $(\underline{\mathbf{a}} + \underline{\mathbf{b}}) \times \underline{\mathbf{c}} = (\underline{\mathbf{a}} \times \underline{\mathbf{c}}) + (\underline{\mathbf{b}} \times \underline{\mathbf{c}})$: distributive
- $\underline{\mathbf{a}} \times \underline{\mathbf{b}} = -\underline{\mathbf{b}} \times \underline{\mathbf{a}}$: NON-commutative
- $(\underline{\mathbf{a}} \times \underline{\mathbf{b}}) \times \underline{\mathbf{c}} \neq \underline{\mathbf{a}} \times (\underline{\mathbf{b}} \times \underline{\mathbf{c}})$: NON-associative
- If m is a scalar,
  $m(\underline{\mathbf{a}} \times \underline{\mathbf{b}}) = (m\underline{\mathbf{a}}) \times \underline{\mathbf{b}} = \underline{\mathbf{a}} \times (m\underline{\mathbf{b}}) = (\underline{\mathbf{a}} \times \underline{\mathbf{b}})m$
- $\underline{\mathbf{a}} \times \underline{\mathbf{b}} = 0$ if vectors are parallel (0^o)
  i.e $\underline{\mathbf{a}} \times \underline{\mathbf{a}} = 0$

Vector product in components

Cross product written out in components:

- $\underline{\mathbf{a}} \times \underline{\mathbf{b}} = (a_x, a_y, a_z) \times (b_x, b_y, b_z)$
  $= (a_x\underline{\mathbf{i}} + a_y\underline{\mathbf{j}} + a_z\underline{\mathbf{k}}) \times (b_x\underline{\mathbf{i}} + b_y\underline{\mathbf{j}} + b_z\underline{\mathbf{k}})$
- Since $\underline{\mathbf{i}} \times \underline{\mathbf{i}} = \underline{\mathbf{j}} \times \underline{\mathbf{j}} = \underline{\mathbf{k}} \times \underline{\mathbf{k}} = 0$ and $\underline{\mathbf{i}} \times \underline{\mathbf{j}} = \underline{\mathbf{k}}$ etc.
- $\underline{\mathbf{a}} \times \underline{\mathbf{b}} = (a_y b_z - a_z b_y)\underline{\mathbf{i}} - (a_x b_z - a_z b_x)\underline{\mathbf{j}} + (a_x b_y - a_y b_x)\underline{\mathbf{k}}$

This is much easier when we write in *determinant* form:

$$\underline{\mathbf{a}} \times \underline{\mathbf{b}} = \begin{vmatrix} \underline{\mathbf{i}} & \underline{\mathbf{j}} & \underline{\mathbf{k}} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} \tag{1}$$

2.1.3 Examples of scalar & vector products in mechanics

- a) Scalar product

  Work done on a body by a force through distance $\underline{\mathbf{dr}}$ from position 1 to 2

  $$W_{12} = \int_1^2 \underline{\mathbf{F}}.\underline{\mathbf{dr}}$$

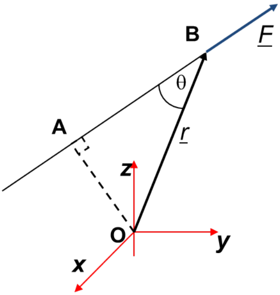
  Only the component of force parallel to the line of displacement does work.

- b) Vector product

  A *torque* about O due to a force $\underline{\mathbf{F}}$ acting at B :

  $$\underline{\tau} = \underline{\mathbf{r}} \times \underline{\mathbf{F}}$$

  Torque is a vector with direction perpendicular to both $\underline{\mathbf{r}}$ and $\underline{\mathbf{F}}$, magnitude of $|\underline{\mathbf{r}}||\underline{\mathbf{F}}| \sin\theta$.



2.2 Differentiation of vectors

Notation: a dot above the function indicates derivative wrt time. A "dash" indicates derivative wrt a spatial coordinate.

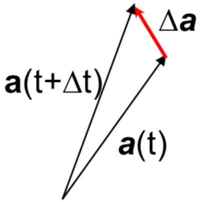
$$\dot{y} \equiv \frac{dy}{dt} \qquad\qquad y' \equiv \frac{dy}{dx}$$

$$\dot{\underline{\mathbf{a}}} = \lim_{\Delta t \to 0} \frac{\underline{\mathbf{a}}(t+\Delta t) - \underline{\mathbf{a}}(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta \underline{\mathbf{a}}}{\Delta t}$$

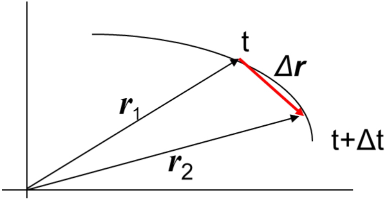
$$\underline{\mathbf{a}}(t) = a_x(t)\underline{\mathbf{i}} + a_y(t)\underline{\mathbf{j}} + a_z(t)\underline{\mathbf{k}}$$

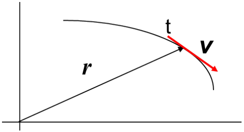
$$\dot{\underline{\mathbf{a}}} = \lim_{\Delta t \to 0} \left(\frac{a_x(t+\Delta t) - a_x(t)}{\Delta t}\, \underline{\mathbf{i}} + \ldots \right)$$

Hence $\dot{\underline{\mathbf{a}}} = \dot{a}_x\underline{\mathbf{i}} + \dot{a}_y\underline{\mathbf{j}} + \dot{a}_z\underline{\mathbf{k}}$



2.2.1 Vector velocity





- $\Delta \underline{\mathbf{r}} = \underline{\mathbf{r}}_2 - \underline{\mathbf{r}}_1$

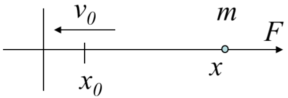
  $$\underline{\mathbf{v}} = \lim_{\Delta t \to 0} \frac{\Delta \underline{\mathbf{r}}}{\Delta t}$$

- Velocity at any point is tangent to the path at that point
- $\underline{\mathbf{v}} = \frac{d\underline{\mathbf{r}}}{dt} = \dot{\underline{\mathbf{r}}}$

In one dimension:
Abandon vector notation and simply write $v = \frac{dx}{dt} = \dot{x}$,
($+v$ in $+x$ direction, $-v$ in $-x$ direction).

Example - 1D motion

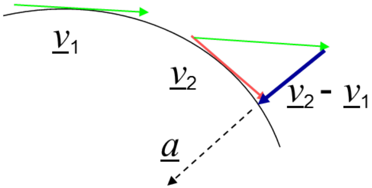
A body has velocity $v_0 = -15\,ms^{-1}$ at position $x_0 = 20m$ and has a time-dependent acceleration $a(t) = 6t - 4\,[ms^{-2}]$. Find the value of x for which the body instantaneously comes to rest.



- $a(t) = 6t - 4\ [ms^{-2}]\ ;\ x_0 = 20\,m\ ;\ v_0 = -15\,ms^{-1}$
- $\dot{v} = 6t - 4 \quad \rightarrow \quad v = \int a(t)dt = 3t^2 - 4t + c$
- At $t = 0, v = -15\ ms^{-1} \quad \rightarrow \quad c = -15\ ms^{-1}$
- $v = 3t^2 - 4t - 15$
- $v = 0$ for $3t^2 - 4t - 15 = 3(t-3)(t+\frac{5}{3}) = 0$

  $\rightarrow t = 3s$ (also $-\frac{5}{3}s$)
- $x = \int v(t)dt = t^3 - 2t^2 - 15t + c' \rightarrow x = 20\,m$ at $t = 0, c' = 20\,m$
- $x(t) = 27 - 18 - 45 + 20 = -16\,\text{m}$

2.2.2 Vector acceleration

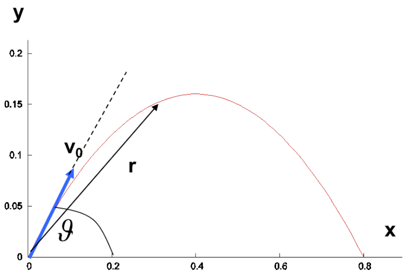


- $\Delta\underline{\mathbf{v}} = \underline{\mathbf{v}}_2 - \underline{\mathbf{v}}_1$

$$\underline{\mathbf{a}} = \lim_{\Delta t \to 0} \frac{\Delta\underline{\mathbf{v}}}{\Delta t} = \dot{\underline{\mathbf{v}}} = \ddot{\underline{\mathbf{r}}}$$

In one dimension:
Abandon vector notation and simply write $a = \frac{dv}{dt} = \dot{v} = \ddot{x}$,
($+a$ in $+x$ direction, $-a$ in $-x$ direction).

Example: constant acceleration - projectile motion in 2D



- $\underline{\mathbf{a}} = \frac{d\underline{\mathbf{v}}}{dt} = \text{constant}$
- $\underline{\mathbf{r}} = 0$ at $t = 0$
- $\int_{v_0}^{v} d\underline{\mathbf{v}} = \int_0^t \underline{\mathbf{a}} dt$

  $\rightarrow \underline{\mathbf{v}} = \underline{\mathbf{v}}_0 + \underline{\mathbf{a}} t \quad \rightarrow \underline{\mathbf{v}} = \frac{d\underline{\mathbf{r}}}{dt}$
- $\int_0^r d\underline{\mathbf{r}} = \int_0^t (\underline{\mathbf{v}}_0 + \underline{\mathbf{a}} t) dt$

  $\rightarrow \underline{\mathbf{r}} = \underline{\mathbf{v}}_0 t + \frac{1}{2} \underline{\mathbf{a}} t^2$

Under gravity: $\underline{\mathbf{a}} = -g\underline{\hat{\mathbf{y}}} \quad \rightarrow \quad a_x = 0; \;\; a_y = -g$

- $v_x = v_0 \cos\theta$
- $v_y = v_0 \sin\theta - gt$

- $x = (v_0 \cos\theta) t$
- $y = (v_0 \sin\theta) t - \frac{1}{2} g t^2$

Trajectory: $\quad y = (\tan\theta) x - \frac{g}{2 v_0^2} (\sec^2\theta) x^2$

The monkey and the hunter