LECTURE 2:

VECTOR MULTIPLICATION -

SCALAR AND VECTOR PRODUCTS

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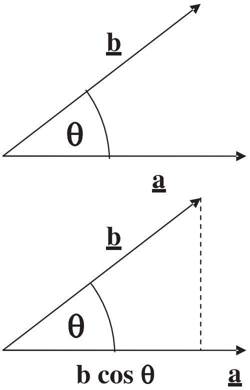
Outline: 2. VECTOR MULTIPLICATION

2.1 Scalar Product

Scalar (or dot) product definition:

$$\underline{\mathbf{a}}.\underline{\mathbf{b}} = |\underline{\mathbf{a}}|.|\underline{\mathbf{b}}| \cos\theta \;\equiv\; ab\cos\theta$$

(write shorthand $|\underline{\mathbf{a}}| = \mathrm{a}$).



- Scalar product is the magnitude of $\underline{\mathbf{a}}$ multiplied by the projection of $\underline{\mathbf{b}}$ onto $\underline{\mathbf{a}}$.
- Obviously if $\underline{\mathbf{a}}$ is perpendicular to $\underline{\mathbf{b}}$ then $\underline{\mathbf{a}}.\underline{\mathbf{b}} = 0$
- Also $\underline{\mathbf{a}}.\underline{\mathbf{a}} = |a|^2$ (since $\theta = 0^\circ$)
  Hence $a = \sqrt{(\underline{\mathbf{a}}.\underline{\mathbf{a}})}$

2.1.1 Properties of scalar product

(i) $\underline{\mathbf{i}}.\underline{\mathbf{i}} = \underline{\mathbf{j}}.\underline{\mathbf{j}} = \underline{\mathbf{k}}.\underline{\mathbf{k}} = 1$ and $\underline{\mathbf{i}}.\underline{\mathbf{j}} = \underline{\mathbf{j}}.\underline{\mathbf{k}} = \underline{\mathbf{k}}.\underline{\mathbf{i}} = 0$

(ii) This leads to $\underline{\mathbf{a}}.\underline{\mathbf{b}} = (a_x\underline{\mathbf{i}} + a_y\underline{\mathbf{j}} + a_z\underline{\mathbf{k}}).(b_x\underline{\mathbf{i}} + b_y\underline{\mathbf{j}} + b_z\underline{\mathbf{k}})$
= $a_x b_x + a_y b_y + a_z b_z$ (this is a very useful relation)

iii) $\underline{\mathbf{a}}.\underline{\mathbf{b}} = \underline{\mathbf{b}}.\underline{\mathbf{a}}$: commutative
$\underline{\mathbf{a}}.(\underline{\mathbf{b}} + \underline{\mathbf{c}}) = \underline{\mathbf{a}}.\underline{\mathbf{b}} + \underline{\mathbf{a}}.\underline{\mathbf{c}}$: distributive

(iv) If $\underline{\mathbf{c}} = \underline{\mathbf{a}} + \underline{\mathbf{b}}$
Then $c^2 = \underline{\mathbf{c}}.\underline{\mathbf{c}} = (\underline{\mathbf{a}} + \underline{\mathbf{b}}).(\underline{\mathbf{a}} + \underline{\mathbf{b}})$
= $a^2 + b^2 + 2\underline{\mathbf{a}}.\underline{\mathbf{b}}$ = $a^2 + b^2 + 2ab\cos(\theta_{ab})$

(v) Parentheses are important
Note $(\underline{\mathbf{u}}.\underline{\mathbf{v}})\ \underline{\mathbf{w}} \neq \underline{\mathbf{u}}\ (\underline{\mathbf{v}}.\underline{\mathbf{w}})$ because one is a vector along $\underline{\hat{\mathbf{w}}}$, the other is along $\underline{\hat{\mathbf{u}}}$.

2.1.2 Angle between two vectors

By definition $\cos(\theta) = \frac{\mathbf{\underline{a}}.\mathbf{\underline{b}}}{ab}$

Example

- The angle between vectors $\underline{\mathbf{a}}$= (3,1,5) and $\underline{\mathbf{b}}$= (2,1,3)
- $\cos\theta = \frac{3\times2+1\times1+5\times3}{\sqrt{(3^2+1^2+5^2)}\times\sqrt{(2^2+1^2+3^2)}} = \frac{22}{\sqrt{(35)}\times\sqrt{(14)}} = 0.994$
- $\theta = 6.3^\circ$

Example of scalar products in physics

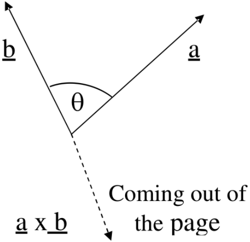
- Work done on a body by a force through distance $\underline{\mathbf{dx}}$
- $dW = \underline{\mathbf{F}}.\underline{\mathbf{dx}}$
- Only the component of force parallel to displacement does work.

2.2 *Vector Product*

Vector (or cross) product of two vectors, definition:

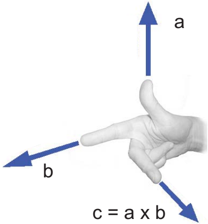
$$\underline{\mathbf{a}} \times \underline{\mathbf{b}} = |\underline{\mathbf{a}}||\underline{\mathbf{b}}| \sin\theta \, \underline{\hat{\mathbf{n}}}$$

where $\underline{\hat{\mathbf{n}}}$ is a *unit vector* in a direction *perpendicular* to both $\underline{\mathbf{a}}$ and $\underline{\mathbf{b}}$.



To get direction of $\underline{\mathbf{a}} \times \underline{\mathbf{b}}$ use right hand rule:

- i) Make a set of directions with your *right hand* $\rightarrow$ thumb & first index finger, and with middle finger positioned perpendicular to plane of both
- ii) Point your thumb along the first vector $\underline{\mathbf{a}}$
- iii) Point your 1st index finger along $\underline{\mathbf{b}}$, making the smallest possible angle to $\underline{\mathbf{a}}$
- iv) The direction of the middle finger gives the direction of $\underline{\mathbf{a}} \times \underline{\mathbf{b}}$.



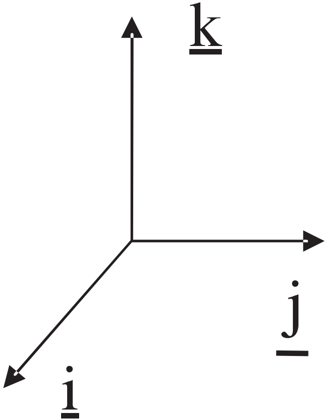
2.2.1 Properties of vector product

- $(\underline{\mathbf{a}} + \underline{\mathbf{b}}) \times \underline{\mathbf{c}} = (\underline{\mathbf{a}} \times \underline{\mathbf{c}}) + (\underline{\mathbf{b}} \times \underline{\mathbf{c}})$: distributive
- $\underline{\mathbf{a}} \times \underline{\mathbf{b}} = -\underline{\mathbf{b}} \times \underline{\mathbf{a}}$: NON-commutative
- $(\underline{\mathbf{a}} \times \underline{\mathbf{b}}) \times \underline{\mathbf{c}} \neq \underline{\mathbf{a}} \times (\underline{\mathbf{b}} \times \underline{\mathbf{c}})$: NON-associative
- If m is a scalar,
  $m(\underline{\mathbf{a}} \times \underline{\mathbf{b}}) = (m\underline{\mathbf{a}}) \times \underline{\mathbf{b}} = \underline{\mathbf{a}} \times (m\underline{\mathbf{b}}) = (\underline{\mathbf{a}} \times \underline{\mathbf{b}})m$.
- Importantly $\underline{\mathbf{a}} \times \underline{\mathbf{b}} = 0$ if vectors are parallel (0^o)
  i.e $\quad \underline{\mathbf{a}} \times \underline{\mathbf{a}} = 0$

2.2.2 Vector product of unit vectors

The basis vectors are connected by cyclic permutations of vector products (another good way to remember the right hand rule)

- $\underline{\mathbf{i}} \times \underline{\mathbf{j}} = \underline{\mathbf{k}}$
- $\underline{\mathbf{j}} \times \underline{\mathbf{k}} = \underline{\mathbf{i}}$
- $\underline{\mathbf{k}} \times \underline{\mathbf{i}} = \underline{\mathbf{j}}$



2.2.3 Vector product in components

A very useful property:

- $\underline{\mathbf{a}} \times \underline{\mathbf{b}} = (a_x, a_y, a_z) \times (b_x, b_y, b_z)$
  $= (a_x\underline{\mathbf{i}} + a_y\underline{\mathbf{j}} + a_z\underline{\mathbf{k}}) \times (b_x\underline{\mathbf{i}} + b_y\underline{\mathbf{j}} + b_z\underline{\mathbf{k}})$
- Since $\underline{\mathbf{i}} \times \underline{\mathbf{i}} = \underline{\mathbf{j}} \times \underline{\mathbf{j}} = \underline{\mathbf{k}} \times \underline{\mathbf{k}} = 0$ and $\underline{\mathbf{i}} \times \underline{\mathbf{j}} = \underline{\mathbf{k}}$ etc.
- $\underline{\mathbf{a}} \times \underline{\mathbf{b}} = (a_y b_z - a_z b_y)\underline{\mathbf{i}} - (a_x b_z - a_z b_x)\underline{\mathbf{j}} + (a_x b_y - a_y b_x)\underline{\mathbf{k}}$

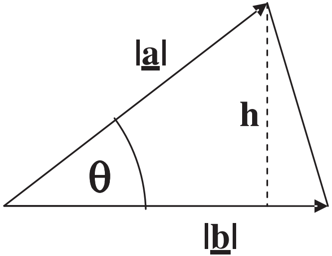
This is much easier when we write in *determinant* form:

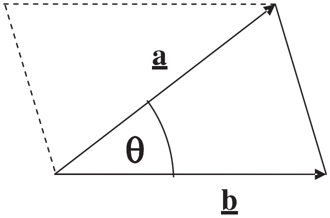
$$\underline{\mathbf{a}} \times \underline{\mathbf{b}} = \begin{vmatrix} \underline{\mathbf{i}} & \underline{\mathbf{j}} & \underline{\mathbf{k}} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} . \qquad (1)$$

2.2.4 Geometrical interpretation of vector product

Vector product is related to the area of a triangle:

- Height of triangle $h = a \sin\theta$
- Area of triangle = A_{triangle} = 1/2 $\times$ base $\times$ height $= \frac{bh}{2} = \frac{ab \sin\theta}{2} = \frac{|\underline{\mathbf{a}} \times \underline{\mathbf{b}}|}{2}$
- Vector product therefore gives the area of the parallelogram: $A_{\text{parallelogram}} = |\underline{\mathbf{a}} \times \underline{\mathbf{b}}|$
- Hence *"vector area"* $\underline{\mathbf{A}}_{\text{parallelogram}} = \underline{\mathbf{a}} \times \underline{\mathbf{b}}$ where the vector points perpendicular to the plane of the parallelogram.





2.3 Examples

Example 1

Find the area of a parallelogram defined by coordinates (0,0,0), (1,3,4) and (2,1,3).

- Make vectors $\underline{\mathbf{a}} = (\underline{\mathbf{i}} + 3\underline{\mathbf{j}} + 4\underline{\mathbf{k}})$ and $\underline{\mathbf{b}} = (2\underline{\mathbf{i}} + \underline{\mathbf{j}} + 3\underline{\mathbf{k}})$

$$\underline{\mathbf{a}} \times \underline{\mathbf{b}} = \begin{vmatrix} \underline{\mathbf{i}} & \underline{\mathbf{j}} & \underline{\mathbf{k}} \\ 1 & 3 & 4 \\ 2 & 1 & 3 \end{vmatrix}. \tag{2}$$

- $\underline{\mathbf{a}} \times \underline{\mathbf{b}} = (3 \times 3 - 4 \times 1)\underline{\mathbf{i}} - (1 \times 3 - 4 \times 2)\underline{\mathbf{j}} + (1 \times 1 - 3 \times 2)\underline{\mathbf{k}}$ = $5\underline{\mathbf{i}} + 5\underline{\mathbf{j}} - 5\underline{\mathbf{k}}$
- Thus the area is $\sqrt{(5^2 + 5^2 + 5^2)} = 8.7$

This method certainly beats 1/2 $\times$ base $\times$ height !

Example 2

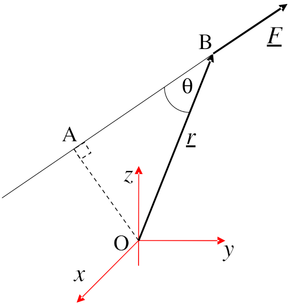
Example of scalars and cross product:
Show that if $\underline{\mathbf{a}} = \underline{\mathbf{b}} + \lambda\underline{\mathbf{c}}$ for some scalar λ, then $\underline{\mathbf{a}} \times \underline{\mathbf{c}} = \underline{\mathbf{b}} \times \underline{\mathbf{c}}$.

- Solution: $\underline{\mathbf{a}} = \underline{\mathbf{b}} + \lambda\underline{\mathbf{c}} \Rightarrow$
  $\underline{\mathbf{a}} \times \underline{\mathbf{c}} = (\underline{\mathbf{b}} + \lambda\underline{\mathbf{c}}) \times \underline{\mathbf{c}} = \underline{\mathbf{b}} \times \underline{\mathbf{c}} + \lambda\underline{\mathbf{c}} \times \underline{\mathbf{c}}$
- but $\underline{\mathbf{c}} \times \underline{\mathbf{c}} = 0$
- so $\underline{\mathbf{a}} \times \underline{\mathbf{c}} = \underline{\mathbf{b}} \times \underline{\mathbf{c}}$ QED

Examples of vector products in Physics

- a) Torque

  A *torque* about O due to a force $\underline{\mathbf{F}}$ acting at B : $\underline{\mathbf{T}} = \underline{\mathbf{r}} \times \underline{\mathbf{F}}$. Torque is a vector with direction perpendicular to both $\underline{\mathbf{r}}$ and $\underline{\mathbf{F}}$, magnitude of $|\underline{\mathbf{r}}||\underline{\mathbf{F}}| \sin\theta$.



- b) Angular momentum

  A body with momentum $\underline{\mathbf{p}}$ at position $\underline{\mathbf{r}}$ has angular momentum about O of $\underline{\mathbf{L}} = \underline{\mathbf{r}} \times \underline{\mathbf{p}}$. Angular momentum is a vector with direction perpendicular to both $\underline{\mathbf{r}}$ and $\underline{\mathbf{p}}$, magnitude of $|\underline{\mathbf{r}}||\underline{\mathbf{p}}| \sin\theta$.

- c) Lorentz force

  The force exerted by a magnetic field $\underline{\mathbf{B}}$ on a charge q moving with velocity $\underline{\mathbf{v}}$ is given by $\underline{\mathbf{F}} = q\underline{\mathbf{v}} \times \underline{\mathbf{B}}$