Classical Mechanics

LECTURE 19:

EXAMPLES OF ELLIPTICAL ORBITS

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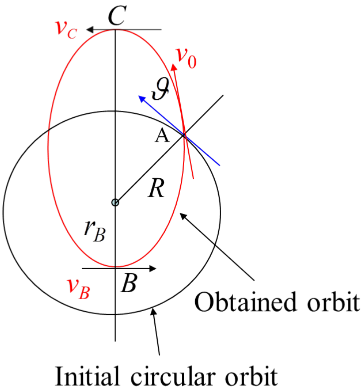
OUTLINE : 19. EXAMPLES OF ELLIPTICAL ORBITS

19.1 Example: mistake in the direction of a satellite

- Mistake is made in boosting a satellite, at radius R, into circular orbit : magnitude of velocity is right but direction is wrong.
- Intended to apply thrust to give velocity v_0 along circular orbit.
- Instead thrust at angle θ wrt direction of motion.
- Energy of orbit is right, angular momentum is wrong.



What is the perigee and apogee of the resulting orbit? (Points B & C)

- Conservation of angular momentum, points A & B

$$J = mv_0 R \sin(\tfrac{\pi}{2} - \theta) = mv_B r_B$$

- Energy at A = energy at perigee B

$$\tfrac{1}{2}mv_0^2 - \tfrac{\alpha}{R} = \tfrac{1}{2}m\dot{r}^2 + \tfrac{J^2}{2mr_B^2} - \tfrac{\alpha}{r_B}$$

where $\alpha = GMm$

19.1 Example continued

- At point B, $\dot{r} = 0$, energy conservation becomes

  $$\frac{1}{2}mv_0^2 - \frac{\alpha}{R} = \frac{m^2 v_0^2 R^2 \cos^2\theta}{2mr_B^2} - \frac{\alpha}{r_B}$$

- Equate forces for circular motion to get v_0 :

  $$\frac{mv_0^2}{R} = \frac{\alpha}{R^2} \quad \rightarrow v_0^2 = \frac{\alpha}{mR}$$

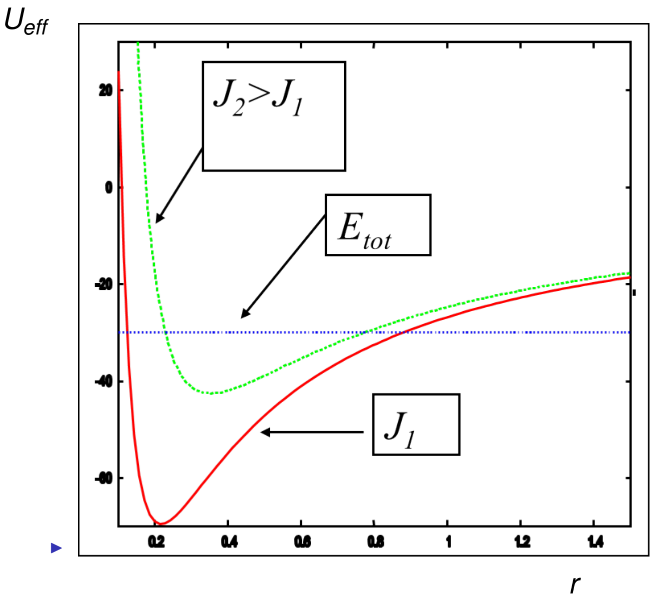
- Sub for v_0^2 : energy conservation becomes

  $$\frac{\alpha}{2R} - \frac{\alpha}{R} = \frac{\alpha R \cos^2\theta}{2r_B^2} - \frac{\alpha}{r_B}$$

- $-\frac{1}{2R} = \frac{R\cos^2\theta}{2r_B^2} - \frac{1}{r_B}$
- $(\times 2Rr_B^2) \quad \rightarrow \quad r_B^2 - 2Rr_B + R^2\cos^2\theta = 0$
- $r_B = R - \sqrt{R^2 - R^2\cos^2\theta}$, also $r_C = R + \sqrt{R^2 - R^2\cos^2\theta}$
- $r_B = R(1 - \sin\theta)$ Perigee

  $r_C = R(1 + \sin\theta)$ Apogee

19.1.1 Orbits with the same energy



19.2 Impulse leaving angular momentum unchanged

- Example: A satellite in circular orbit has been given an impulse leaving J unchanged. The kinetic energy is changed by $T = \beta T_0$. Describe the subsequent motion.
- If J is not changed, impulse must be perpendicular to the direction of motion, with angular part of the velocity unchanged.

- $E = \frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} - \frac{\alpha}{r}$
- Circular orbit:

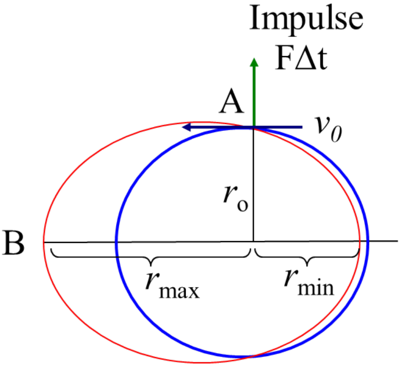
$\rightarrow \dot{r} = 0\ ,\ J = mr_0 v_0 \quad (1)$

$\rightarrow E_{initial} = \frac{1}{2}mv_0^2 - \frac{\alpha}{r_0}$

- Equate forces :

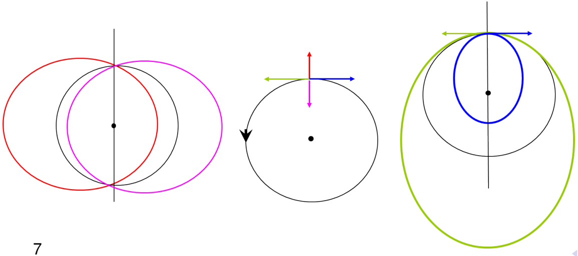
$\rightarrow \frac{mv_0^2}{r_0} = \frac{\alpha}{r_0^2}$

$\rightarrow v_0^2 = \frac{\alpha}{mr_0} \qquad (2)$



19.2 Example continued

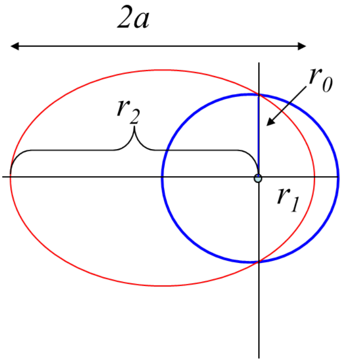
- New orbit (elliptical): $E_{new} = \frac{1}{2}\beta m v_0^2 - \frac{\alpha}{r_0}$
- Equate energies: subsequent motion described by:
- $\frac{1}{2}\beta m v_0^2 - \frac{\alpha}{r_0} = \frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} - \frac{\alpha}{r}$ (3)
- Now solve for r_{min} , r_{max} . Set $\dot{r} = 0$
- From (1), (2), (3) $\rightarrow$ $(\beta - 2)\, r^2 + 2r_0\, r - r_0^2 = 0$
- $r_{min,max} = \frac{-r_0 \pm \sqrt{r_0^2 + (\beta-2)\, r_0^2}}{(\beta-2)}$
- Example: $\beta = 1.001 \rightarrow r_{max} = 1.033\, r_0$ $r_{min} = 0.968\, r_0$

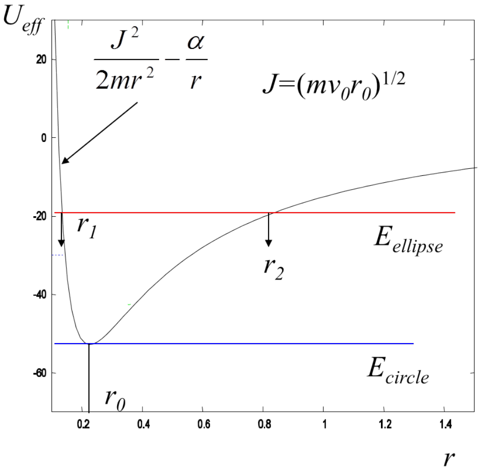


Changes in orbit as a result of impulse

19.2.1 Orbits with the same angular momentum

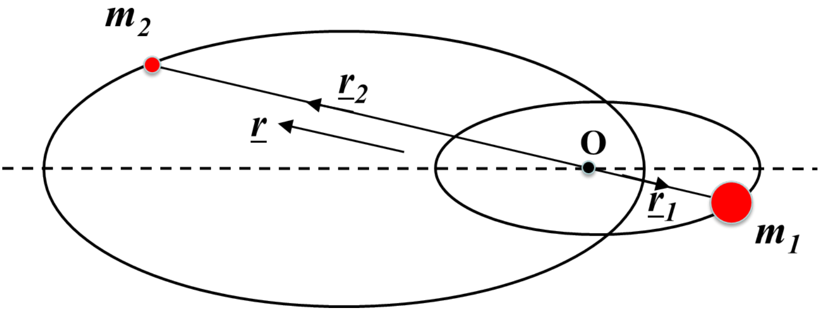
- $E_{ellipse} = -\frac{\alpha}{2\,a}$
- $E_{circle} = -\frac{\alpha}{2\,r_0}$





19.3 Mutual orbits

- The two bodies make a mutual elliptical orbit on either side of the C of M (origin) in a straight line through the C of M
- Relative position vector : $\underline{\mathbf{r}} = \underline{\mathbf{r_2}} - \underline{\mathbf{r_1}}$
- Definition of C of M about O : $m_1 \underline{\mathbf{r_1}} + m_2 \underline{\mathbf{r_2}} = 0$



Mutual orbits continued

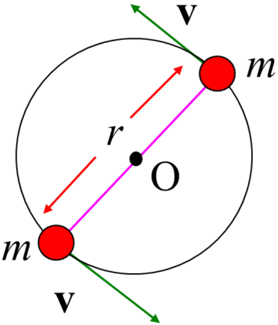
- Internal forces: $\underline{\mathbf{F}}_{\mathbf{12}} = m_1 \ddot{\underline{\mathbf{r}}}_{\mathbf{1}}$, $\underline{\mathbf{F}}_{\mathbf{21}} = m_2 \ddot{\underline{\mathbf{r}}}_{\mathbf{2}}$
  Then $\ddot{\underline{\mathbf{r}}} = \ddot{\underline{\mathbf{r}}}_{\mathbf{2}} - \ddot{\underline{\mathbf{r}}}_{\mathbf{1}} = \frac{\underline{\mathbf{F}}_{\mathbf{21}}}{m_2} - \frac{\underline{\mathbf{F}}_{\mathbf{12}}}{m_1}$
  But $\underline{\mathbf{F}}_{\mathbf{12}} = -\underline{\mathbf{F}}_{\mathbf{21}}$
- Hence $\ddot{\underline{\mathbf{r}}} = \underline{\mathbf{F}}_{\mathbf{21}} \left(\frac{1}{m_1} + \frac{1}{m_2} \right)$
- Define $\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2} \quad \rightarrow \quad \mu = \frac{m_1 m_2}{m_1 + m_2}$
  μ is the *reduced mass of the system*

- Hence $\mu \, \ddot{\underline{\mathbf{r}}} = \underline{\mathbf{F}}_{\mathbf{21}}$
  and $\mu \, \ddot{\underline{\mathbf{r}}} = -\frac{G m_1 m_2}{|\underline{\mathbf{r}}_{\mathbf{2}} - \underline{\mathbf{r}}_{\mathbf{1}}|^2} \, \hat{\underline{\mathbf{r}}} = -\frac{G \mu \, (m_1 + m_2)}{|\underline{\mathbf{r}}_{\mathbf{2}} - \underline{\mathbf{r}}_{\mathbf{1}}|^2} \, \hat{\underline{\mathbf{r}}}$

- Therefore Newton's Second Law for mutual motion can be re-written in terms of the position of the second body with respect to the first. The second body has the reduced mass which orbits round the first body with an effective mass equal to the sum of the two masses.

19.3.1 Example: binary star

A binary star consists of two stars bound together by gravity moving in roughly opposite directions along a nearly circular orbit. The period of revolution of the starts about their centre of mass is 14.4 days and the speed of each component is 220 km s^{-1}. Find the distance between the two stars and their masses.



- For single star : $v = (\frac{r}{2})\,\omega = \frac{r}{2}\frac{2\pi}{T}$
- $r = \frac{vT}{\pi} = 8.7 \times 10^{10}$ m
- Mutual motion : $\mu\,(\ddot{r} - r\,\dot{\theta}^2)\underline{\hat{\mathbf{r}}} = -\frac{Gm_1 m_2}{r^2}\underline{\hat{\mathbf{r}}}$
- For circular motion :

  $\ddot{r} = \dot{r} = 0$, $r\,\dot{\theta}^2$ = constant $= r\omega^2$
- Equating forces :

  $r\mu\omega^2 = \frac{Gm_1 m_2}{r^2} = \frac{G\mu(m_1+m_2)}{r^2}$
- $(m_1 + m_2) = \frac{r^3\omega^2}{G}$; $m_1 = m_2$ (symmetry)
- $m_1 = m_2 = 1.25 \times 10^{32}$ kg