Classical Mechanics

LECTURE 18:

CENTRAL FORCE : THE ORBIT EQUATION

Prof. N. Harnew
University of Oxford

HT 2017

OUTLINE : 18. CENTRAL FORCE: THE ORBIT EQUATION

18.1 The orbit equation

Note that the derivation of this is off syllabus

- Acceleration in polar coordinates

  $\underline{\mathbf{a}} = \underline{\ddot{\mathbf{r}}} = (\ddot{r} - r\,\dot{\theta}^2)\,\underline{\hat{\mathbf{r}}} + \frac{1}{r}\,\frac{d}{dt}(r^2\dot{\theta})\,\underline{\hat{\theta}}$

- If $\underline{\mathbf{F}} = f(r)\,\underline{\hat{\mathbf{r}}}$ only, then $F_\theta = 0$. $J = mr^2\dot{\theta} =$ constant.

  $\rightarrow\ F_r = m(\ddot{r} - r\,\dot{\theta}^2) = -\frac{\alpha}{r^2}$ (gravitational force, $\alpha = GmM$)

- Hence $\ddot{r} = \frac{J^2}{m^2\,r^3} - \frac{\alpha}{mr^2} = \frac{u^3\,J^2}{m^2} - \frac{u^2\,\alpha}{m}$ (where $u = \frac{1}{r}$) (1)
- $\dot{r} = \frac{d\theta}{dt}\frac{dr}{d\theta} = \frac{J}{m\,r^2}\frac{dr}{d\theta} = -\frac{J}{m}\frac{d(1/r)}{d\theta} = -\frac{J}{m}\frac{du}{d\theta}$
- $\ddot{r} = \frac{d}{dt}(\dot{r}) = \frac{d\theta}{dt}\frac{d}{d\theta}\left(-\frac{J}{m}\frac{du}{d\theta}\right) = -(\frac{J^2}{m^2}u^2)\frac{d^2u}{d\theta^2}$
- Substituting in Eq (1) : $-(\frac{J^2}{m^2})u^2\frac{d^2u}{d\theta^2} = \frac{u^3\,J^2}{m^2} - \frac{u^2\,\alpha}{m}$

  $\rightarrow\ \frac{d^2u}{d\theta^2} = -u + \frac{m\,\alpha}{J^2}\ \rightarrow\ \frac{d^2u}{d\theta^2} = -u + \frac{1}{r_0}$ $\quad (r_0 = \frac{J^2}{m\,\alpha})$

The orbit equation continued

$$\frac{d^2u}{d\theta^2} = -u + \frac{1}{r_0}$$ where $u = \frac{1}{r}$ and $$r_0 = \frac{J^2}{m\,\alpha}$$

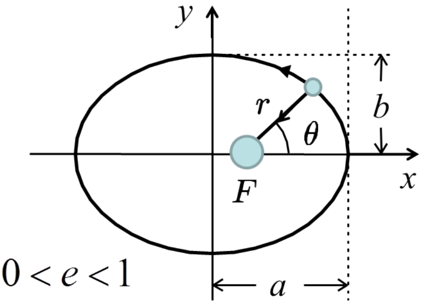
- Solution is $\frac{1}{r} = \frac{1}{r_0} + C\cos(\theta - \theta_0)$ where C, θ_0 = constants

$$r(\theta) = \frac{r_0}{1+e\cos(\theta-\theta_0)}$$ e = eccentricity $\quad (e = C\,r_0)$

- This is in the form of an ellipse†. Also have a link between angular momentum and the ellipse geometry ($J^2 = m\,\alpha r_0$).

† More precisely a conic section, which includes hyperbola, parabola and circle.

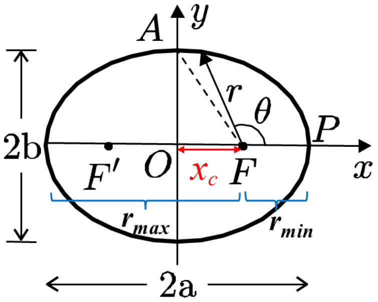
- Choose major axis as x axis $\rightarrow \theta_0 = 0$
- $r(\theta) = \frac{r_0}{1+e\cos\theta}$
- Equivalent form of ellipse : $(\frac{x}{a})^2 + (\frac{y}{b})^2 = 1$
- $b = a\sqrt{(1-e^2)}$
- $a = \frac{r_0}{(1-e^2)}$



18.1.1 The ellipse geometry

Example of a rotating planet : the sun is at the ellipse focus F

$$r(\theta) = \frac{r_0}{1+e\cos\theta} \qquad \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$



- Closest approach $\theta = 0$
  "perigee" $r_{min} = \frac{r_0}{1+e}$
  Furthest approach $\theta = \pi$
  "apogee" $r_{max} = \frac{r_0}{1-e}$
  ($\dot{r} = 0$ in both cases)
- $r_{max} + r_{min} = 2a = \frac{2r_0}{1-e^2}$
  $\rightarrow$ $r_0 = a(1-e^2)$
  - $x_c + r_{min} = a \rightarrow x_c + \frac{r_0}{1+e} = a$
    $\rightarrow \quad x_c = a - a(1-e) = a\,e$
  - At point A, $r^2 = x_c^2 + b^2$; $\cos\theta = -\frac{x_c}{r}$; $r = \frac{r_0}{1-ex_c/r}$
    $\rightarrow (r_0 + ex_c)^2 = x_c^2 + b^2 \rightarrow \left(a(1-e^2) + e^2 a\right)^2 = e^2a^2 + b^2$

$b = a\sqrt{(1-e^2)}$ also $r_{min} = a(1-e)$ & $r_{max} = a(1+e)$

18.2 Kepler's Laws

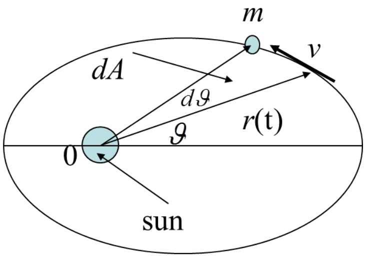
- KI: "The orbit of every planet is an ellipse with the sun at one of the foci".
  [Already derived]

- KII: "A line joining a planet and the sun sweeps out equal areas during equal intervals of time". [Already derived]

- KIII: "The squares of the orbital periods of planets are directly proportional to the cubes of the semi-major axis of the orbits".

18.2.1 Kepler III

"The squares of the orbital periods of planets are directly proportional to the cubes of the semi-major axis of the orbits"



This is trivial for a circle :

$$\rightarrow \; mr_0\omega^2 = mr_0(\tfrac{2\pi}{T})^2 = \frac{GmM}{r_0^2}$$

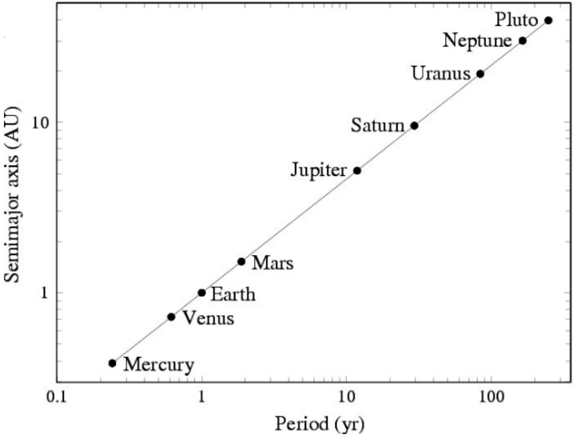
$$\rightarrow r_0^3 = T^2 \frac{GM}{4\pi^2}$$

For an ellipse:

- $\frac{dA}{dt} = \frac{1}{2} r^2 \dot{\theta} = \frac{J}{2m} =$ constant
- Integrate $A = \int_0^T \frac{J}{2m} dt \;\rightarrow\; A^2 = (\frac{J}{2m})^2 T^2$
- From before : $r_0 = a(1-e^2)$, $b = a\sqrt{(1-e^2)} \rightarrow a = \frac{b^2}{r_0}$
- Area of an ellipse : $A = \pi ab \;\rightarrow A^2 = \pi^2 a^3 r_0$
- Putting it all together $\rightarrow \; T^2 \propto a^3$

18.2.2 Planetary data

Kepler-III "The squares of the orbital periods of planets are directly proportional to the cubes of the semi-major axis of the orbits"



- IMPRESSIVE !

18.3 Elliptical orbit via energy ($E_{min} < E < 0$)

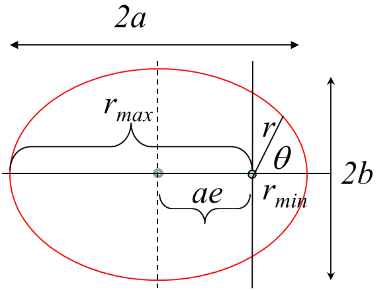
- $E = \frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} - \frac{\alpha}{r}$
- At turning points
  $\dot{r} = 0 \rightarrow r = r_{min}$ or $r = r_{max}$
- $E = \frac{J^2}{2mr^2} - \frac{\alpha}{r}$

  $\rightarrow r^2 + \frac{\alpha}{E} r - \frac{J^2}{2mE} = 0$

  $\rightarrow r = -\frac{\alpha}{2E} \pm \left[\left(\frac{\alpha}{2E}\right)^2 + \frac{J^2}{2mE}\right]^{\frac{1}{2}}$
- $r_{min,max} = -\left(\frac{\alpha}{2E}\right)\left[1 \pm \underbrace{\left(1 + \frac{2EJ^2}{m\alpha^2}\right)^{\frac{1}{2}}}_{e}\right]$

$$r_{max} = \underbrace{-\frac{\alpha}{2E}(1+e)}_{= a(1 + e)} \quad , \quad r_{min} = \underbrace{-\frac{\alpha}{2E}(1-e)}_{= a(1 - e)}$$

Consistent with the orbit equations. NICE!



Also:

$\left(1 + \frac{2EJ^2}{m\alpha^2}\right)^{\frac{1}{2}} = e$

$E = \frac{m\alpha^2}{2J^2}(e^2 - 1)$

$\quad = \frac{\alpha}{2r_0}(e^2 - 1)$

where $r_0 = \frac{J^2}{m\alpha}$ and

e = eccentricity of ellipse.

Elliptical orbit via energy, continued

$$r(\theta) = \frac{r_0}{1+e\cos\theta}$$

Total energy in ellipse parameters

$$E = \frac{\alpha}{2r_0}\left(e^2 - 1\right)$$

- $e = 0, r = r_0, E = -\frac{\alpha}{2r_0}$
  $\rightarrow$ motion in a circle
- If $0 < e < 1\ ,\ E < 0$
  $\rightarrow$ motion is an ellipse
- If $e = 1\ ,\ E = 0$
  $r(\theta) = \frac{r_0}{1+\cos\theta}$
  $\rightarrow$ motion is a parabola
- If $e > 1\ ,\ E > 0$
  $\rightarrow$ motion is a hyperbola