Classical Mechanics

LECTURE 17:

EFFECTIVE POTENTIAL & SIMPLE EXAMPLES

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OUTLINE : 17. EFFECTIVE POTENTIAL & SIMPLE EXAMPLES

17.1 Effective potential

- Energy equation : $E = \frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} + U(r)$
- Define *effective potential* : $U_{eff}(r) = \frac{J^2}{2mr^2} + U(r)$

  $\rightarrow$ then $E = \frac{1}{2}m\dot{r}^2 + U_{eff}(r)$
- Note this has the same form as a 1-D energy expression :

  $\rightarrow$ $E = \frac{1}{2}m\dot{x}^2 + U(x)$

  $\rightarrow$ the analysis becomes 1-D-like problem since $J = \text{const}$
- Allows to predict important features of motion without solving the radial equation

  $\rightarrow$ $\frac{1}{2}m\dot{r}^2 = E - U_{eff}(r)$ $\leftarrow$ LHS is always positive

  $\rightarrow$ $U_{eff}(r) < E$

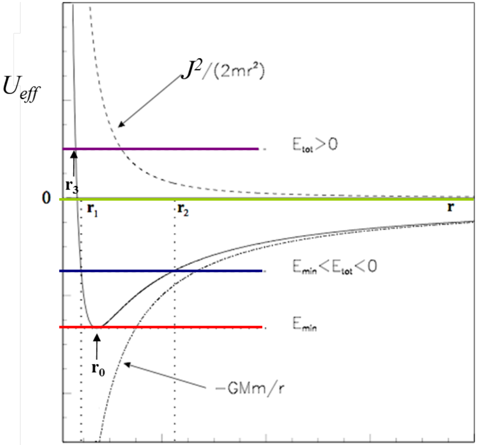
  The only locations where the particle is allowed to go are those with $U_{eff}(r) < E$

17.1.1 $U_{eff}(r)$ for inverse square law

- $U_{eff}(r) = \frac{J^2}{2mr^2} - \frac{GmM}{r}$
- $U_{eff}(r) < E_{tot}$ for all r

Three cases :

- $E_{tot} < 0$: Bound (closed) orbit with $r_1 < r < r_2$
- E_{tot} has minimum energy at $r = r_0$: $\frac{dU_{eff}}{dr} = 0$, circular motion with $\dot{r} = 0$
- $E_{tot} > 0$: Unbound (open) orbit with $r > r_3$



17.2 Examples

17.2.1 Example 1 : 2-D harmonic oscillator

- $\underline{\mathbf{F}} = -k\underline{\mathbf{r}}$ (ignore the natural length of the spring)

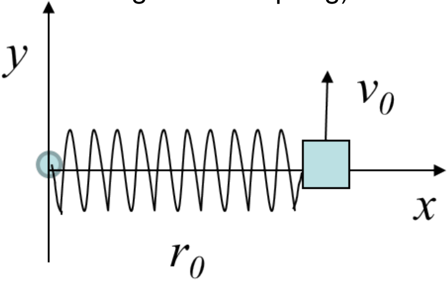
- Energy equation : $E = \frac{1}{2}m\dot{r}^2 + U_{eff}(r)$
- $U_{eff}(r) = \frac{J^2}{2mr^2} + \frac{1}{2}kr^2$
- For circular motion $\dot{r} = 0$:

  E_{min} when $\frac{\partial U_{eff}}{\partial r}|_{r_0} = 0$

  $\rightarrow -\frac{J^2}{mr_0^3} + kr_0 = 0$

  where $J = mv_0 r_0$
- Leads to $\frac{mv_0^2}{r_0} = k\, r_0$

  as expected



Including the natural length:

- $\underline{\mathbf{F}} = -k\underline{\mathbf{r}} \rightarrow \underline{\mathbf{F}} = -k(\underline{\mathbf{r}} - \underline{\mathbf{a}})$
- $U = \frac{1}{2}kr^2 \rightarrow U = \frac{1}{2}k(r-a)^2$

Leads to $\frac{mv_0^2}{r_0} = k\,(r_0 - a)$

Example continued

- $U_{eff}(r) = \frac{J^2}{2mr^2} + \frac{1}{2}kr^2$
- For general motion :
- $\underline{\mathbf{F}} = -k\underline{\mathbf{r}}$

  $\rightarrow \quad m\ddot{x} = -kx$

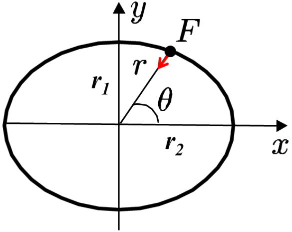
  $\rightarrow \quad m\ddot{y} = -ky$
- Solution for B.C's at $t = 0$:

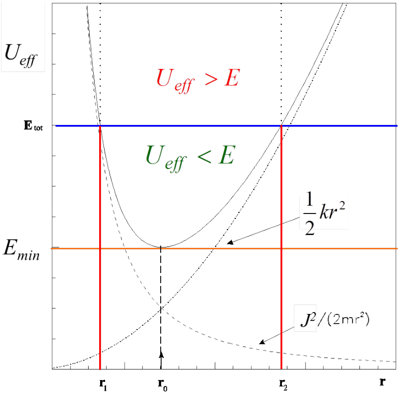
  $x = r_2, y = 0, \dot{x} = 0$

  $\rightarrow \quad x = r_2 \cos \omega t$

  $\rightarrow \quad y = r_1 \sin \omega t$

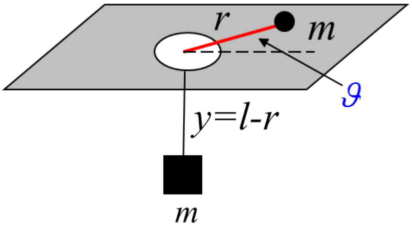
  where $\omega^2 = \frac{k}{m}$
- Ellipse: $\quad (\frac{x}{r_2})^2 + (\frac{y}{r_1})^2 = 1$





17.2.2 Example 2 : Rotating ball on table

Two particles of mass m are connected by a light inextensible string of length ℓ. The particle on the table starts at $t = 0$ at a distance $\ell/2$ from the hole at a speed v_0 perpendicular to the string. Find the speed at which the particle below the table falls.



- Energy equation : $E = \frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} + U(r)$
- $\dot{y} = \dot{r}$, $U(r) = -mgy$
- $E = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}m\dot{y}^2 + \frac{J^2}{2mr^2} - mg(\ell - r)$
  - At $t = 0$: $J = \frac{mv_0\ell}{2}$, $E = \frac{1}{2}mv_0^2 - mg\frac{\ell}{2}$
  - Solution : $\dot{r}^2 = \frac{g\ell+v_0^2}{2} - \frac{(\ell v_0)^2}{8}\frac{1}{r^2} - gr$.

Condition for the particle on the table to move in circular motion

$\rightarrow$ $\dot{r} = 0$, Equate forces $\frac{mv_0^2}{\ell/2} = mg$ $\rightarrow$ gives $\frac{v_0^2}{g\ell} = \frac{1}{2}$

Example 2 continued : effective potential

- Effective potential : $U_{eff} = \frac{J^2}{2mr^2} - mg(\ell - r)$
- Closed orbit with $r_{min} < r < \ell/2$
- Ball never passes though hole in absence of friction, minimum radius $r = r_{min}$