Classical Mechanics

LECTURE 16:

ORBITS : CENTRAL FORCES

Prof. N. Harnew
University of Oxford

HT 2017

OUTLINE : 16. ORBITS : CENTRAL FORCES

16.1 Central force : the equation of motion

- Recall the acceleration in polar coordinates

  $\underline{\mathbf{a}} = \underline{\ddot{\mathbf{r}}} = (\ddot{r} - r\,\dot{\theta}^2)\,\underline{\hat{\mathbf{r}}} + (2\dot{r}\,\dot{\theta} + r\,\ddot{\theta})\,\underline{\hat{\theta}}$

- If $\underline{\mathbf{F}} = f(r)\,\underline{\hat{\mathbf{r}}}$ only, then $F_\theta = 0$

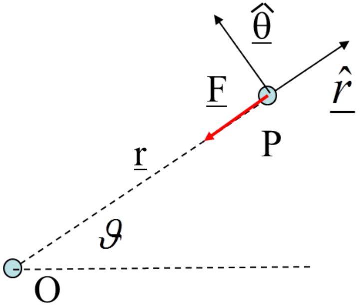
  $\rightarrow\ F_\theta = m(2\dot{r}\,\dot{\theta} + r\,\ddot{\theta}) = 0$

  $\rightarrow\ F_r = m(\ddot{r} - r\,\dot{\theta}^2) = f(r)$

- Consider $\frac{d}{dt}(r^2\dot{\theta}) = 2r\dot{r}\,\dot{\theta} + r^2\,\ddot{\theta}$

  Hence $\frac{1}{r}\,\frac{d}{dt}(r^2\dot{\theta}) = 0$

  $\rightarrow\ (r^2\dot{\theta}) \ = \ $ constant of motion



- The angular momentum in the plane :

  $\underline{\mathbf{J}} = m\,\underline{\mathbf{r}} \times \underline{\mathbf{v}} = m\,\underline{\mathbf{r}} \times (\dot{r}\,\underline{\hat{\mathbf{r}}} + r\,\dot{\theta}\,\underline{\hat{\theta}}) = (mr^2\,\dot{\theta})\,\underline{\hat{\mathbf{n}}}$

  where $\underline{\mathbf{r}} \times \underline{\hat{\mathbf{r}}} = 0$ and $\underline{\hat{\mathbf{n}}} = \underline{\hat{\mathbf{r}}} \times \underline{\hat{\theta}}$

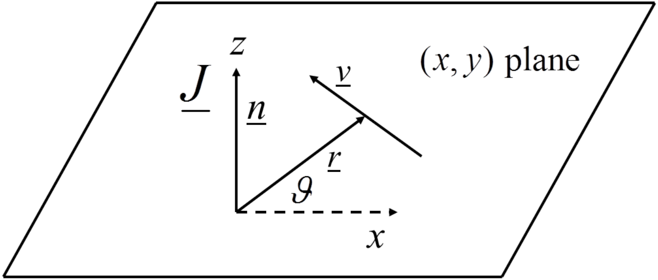
- Torque about origin : $\underline{\tau} = \frac{d\mathbf{J}}{dt} = \underline{\mathbf{r}} \times \underline{\mathbf{F}} = 0$ ($\underline{\mathbf{F}}$ acts along $\underline{\mathbf{r}}$)

  Angular momentum vector is a constant of the motion

16.2 Motion under a central force

16.2.1 Motion in a plane

- $\underline{\mathbf{J}} = m\,\underline{\mathbf{r}} \times \underline{\mathbf{v}}$
- Angular momentum is *always* perpendicular to $\underline{\mathbf{r}}$ and $\underline{\mathbf{v}}$
- $\underline{\mathbf{J}}$ is a constant vector ; $\underline{\mathbf{J}} \cdot \underline{\mathbf{r}} = 0$; $\underline{\mathbf{J}} \cdot \underline{\mathbf{v}} = 0$

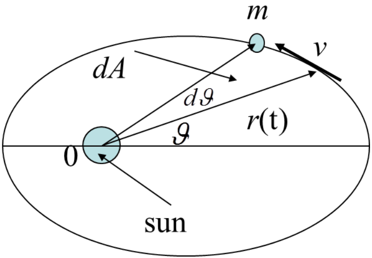


Motion under a central force lies in a plane

16.2.2 Sweeping out equal area in equal time

- Central force example : planetary motion : $|\underline{\mathbf{F}}_r| = \frac{GMm}{r^2}$
- Angular momentum is conserved

  $\rightarrow \ |\underline{\mathbf{J}}| = mr^2\,\dot{\theta} = \text{constant}$



- $dA \approx \frac{1}{2}\, r^2\, d\theta$
- $\frac{dA}{dt} = \frac{1}{2}\, r^2\, \dot{\theta}$

  $\frac{dA}{dt} = \frac{J}{2m} = \text{constant}$ (Kepler 2^{nd} Law)

Orbit sweeps out equal area in equal time

16.3 Central force : the total energy

- Total energy = kinetic + potential :

  $E = T + U(r) = \frac{1}{2}mv^2 + U(r)$ = constant
- $\underline{\mathbf{v}} = \dot{r}\underline{\hat{\mathbf{r}}} + r\dot{\theta}\underline{\hat{\theta}} \rightarrow \ |\underline{\mathbf{v}}|^2 = (\underline{\hat{\mathbf{r}}} + r\dot{\theta}\underline{\hat{\theta}}) \cdot (\underline{\hat{\mathbf{r}}} + r\dot{\theta}\underline{\hat{\theta}})$

  $\rightarrow \ |\underline{\mathbf{v}}|^2 = \dot{r}^2 + r^2\dot{\theta}^2$ (since $\underline{\hat{\mathbf{r}}} \cdot \underline{\hat{\theta}} = 0$)
- $E = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + U(r)$
- No external torque: angular momentum is conserved

  $\rightarrow \ |\underline{\mathbf{J}}| = mr^2\,\dot{\theta} =$ constant

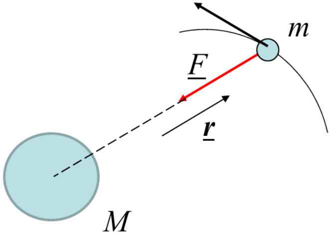
$$E = \frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} + U(r)$$

- Potential energy for a central force

  $U(r) = -\int_{r_{ref}}^{r} \underline{\mathbf{F}} \cdot d\underline{\mathbf{r}} = -\int_{r_{ref}}^{r} f(r)\,dr$

16.3.1 The potential term (inverse square interaction)

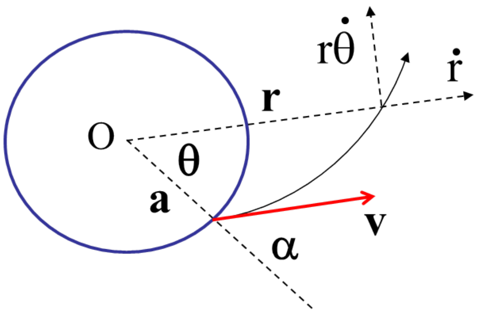
- $\underline{\mathbf{F}} = -\frac{A}{r^2}\,\underline{\hat{\mathbf{r}}} \;\rightarrow\; f(r) = -\frac{A}{r^2}$
  [Attractive force for $A > 0 \;\rightarrow$ signs are important !]
- $U(r) = -\int_{r_{ref}}^{r} \underline{\mathbf{F}} \cdot d\underline{\mathbf{r}}$
  $= -\int_{r_{ref}}^{r} f(r)\,dr$
- $U(r) = -\frac{A}{r} + \frac{A}{r_{ref}}$
  Usual to define $U(r) = 0$ at $r_{ref} = \infty$
  $\rightarrow\; U(r) = -\frac{A}{r}$



Newton law of gravitation : $\underline{\mathbf{F}} = -\frac{GMm}{r^2}\,\underline{\hat{\mathbf{r}}} \;\rightarrow\; U(r) = -\frac{GMm}{r}$

16.3.2 Example

A projectile is fired from the earth's surface with speed *v* at an angle α to the radius vector at the point of launch. Calculate the projectile's subsequent maximum distance from the earth's surface. Assume that the earth is stationary and its radius is *a*.



16.3.2 Example : solution

- $U(r) = -\frac{GMm}{r}$
- $|\underline{\mathbf{J}}| = m|\underline{\mathbf{r}} \times \underline{\mathbf{v}}| = mav\sin\alpha$
- Energy equation : $E = \frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} + U(r)$

  $\rightarrow \; E = \frac{1}{2}m\dot{r}^2 + \frac{ma^2v^2\sin^2\alpha}{2r^2} - \frac{GMm}{r}$
- At $r = a$: $E = \frac{1}{2}mv^2 - \frac{GMm}{a}$. At maximum height : $\dot{r} = 0$

  $\rightarrow \; \frac{1}{2}mv^2 - \frac{GMm}{a} = \frac{ma^2v^2\sin^2\alpha}{2r_{max}^2} - \frac{GMm}{r_{max}} \qquad (1)$

  $\rightarrow \; \left(v^2 - \frac{2GM}{a}\right) r_{max}^2 + 2GM\, r_{max} - a^2v^2\sin^2\alpha = 0$
- Solve and take the positive root
- Note from Equ.(1) : When $\dot{r} \rightarrow 0$ as $r_{max} \rightarrow \infty$, the rocket *just* escapes the earth's gravitational field

  i.e. $\frac{1}{2}mv^2 - \frac{GMm}{a} \rightarrow 0$, $v_{esc} = \sqrt{\frac{2GM}{a}}$ (independent of α)