LECTURE 16:

DIAGONALIZATION OF MATRICES

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Outline: 16. DIAGONALIZATION OF MATRICES

16.1 Similarity transformation

- Consider:
  $|\mathbf{p}\rangle$ & $|\mathbf{q}\rangle$ $\rightarrow$ two general vectors in basis $|\mathbf{e}\rangle$.
  $|\mathbf{p}'\rangle$ & $|\mathbf{q}'\rangle$ $\rightarrow$ the same two vectors transformed to basis $|\mathbf{e}'\rangle$.
- If transformation matrix S transforms $p' \rightarrow p$ and $q' \rightarrow q$
  $\rightarrow$ $p = Sp'$ $\rightarrow$ Equ (1)
  $q = Sq'$ $\rightarrow$ Equ (2)
- If A is a linear operator in basis $|\mathbf{e}\rangle$ which transforms $p \rightarrow q$
  $q = Ap$ $\rightarrow$ Equ (3)
  and operator A' in basis $|\mathbf{e}'\rangle$ $q' = A'p'$ $\rightarrow$ Equ (4)
  From 1) and 3) $q = ASp'$ and then from 2) $Sq' = ASp'$
  Hence $q' = S^{-1}ASp'$ $\rightarrow$ Equ (5)
- Compare 4) and 5) $A' = S^{-1}AS$
- Hence similarity transformation $S^{-1}AS$ represents the transformation of operator A in basis $|\mathbf{e}\rangle$ to the equivalent operator A' in basis $|\mathbf{e}'\rangle$

16.2 Diagonalization of matrices

- Consider a linear operator A in basis $|\mathrm{e}\rangle$. This has eigenvectors/values x_j, λ_j. This is represented in matrix form:
  $Ax_j = \lambda_j x_j$
- Consider a *similarity transformation* into some basis $|\mathrm{e}'\rangle$
  $A \to A' = S^{-1}AS$, where the columns j of the matrix S are the special case of the *eigenvectors* of the matrix A, $\begin{pmatrix} \uparrow & \uparrow & \cdots \\ x_1 & x_2 & \cdots \\ \downarrow & \downarrow & \cdots \end{pmatrix}$
  i.e. $S_{ij} \equiv (x_j)_i$ (for the i^{th} component of x_j).
- Consider the individual elements of $S^{-1}AS$ in this case

$$
\begin{aligned}
A'_{ij} &= (S^{-1}AS)_{ij} \\
&= \sum_k (S^{-1})_{ik} (\sum_m A_{km} S_{mj}) = \sum_k \sum_m (S^{-1})_{ik} A_{km} S_{mj} \\
&= \sum_k \sum_m (S^{-1})_{ik} A_{km} (x_j)_m \\
&= \sum_k (S^{-1})_{ik} \lambda_j (x_j)_k \\
&= \sum_k \lambda_j (S^{-1})_{ik} S_{kj} = \lambda_j \delta_{ij}
\end{aligned}
$$

where δ_{ij} is the Kronecker delta.

Hence $S^{-1}AS$ is a diagonal matrix with the eigenvalues of A along the diagonal.

16.2.1 Prescription for diagonalization of a matrix

To “diagonalize” a matrix:

- Take a given $N \times N$ matrix A
- Construct a matrix S that has the *eigenvectors* of A as its columns
- Then the matrix $(S^{-1}AS)$ is diagonal and has the *eigenvalues* of A as its diagonal elements.
- (Note the diagonal matrix will always be *real* if A is Hermitian.)

16.3 Example of matrix diagonalization

- Diagonalize the matrix

$$A = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 1 & -3 \\ 3 & -3 & -3 \end{pmatrix} \tag{1}$$

- Diagonalize using a matrix of the form $(S^{-1}AS)$
  We already found the orthogonal normalized eigenvectors $\rightarrow$ construct S from the eigenvectors:

$$S = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} \end{pmatrix} = \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{3} & \sqrt{2} & 1 \\ \sqrt{3} & -\sqrt{2} & -1 \\ 0 & \sqrt{2} & -2 \end{pmatrix} \tag{2}$$

- Take the inverse of $S \rightarrow \; S^{-1}$.

$$S^{-1} = \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{3} & \sqrt{3} & 0 \\ \sqrt{2} & -\sqrt{2} & \sqrt{2} \\ 1 & -1 & -2 \end{pmatrix} \tag{3}$$

  Quote without proof: When a matrix is made up of columns of eigenvectors which form an orthonormal set, then
  $\rightarrow \quad S^{-1} = S^{\dagger} = S^{*T}$

- Hence

$$S^{-1}AS = \frac{1}{6}\begin{pmatrix} \sqrt{3} & \sqrt{3} & 0 \\ \sqrt{2} & -\sqrt{2} & \sqrt{2} \\ 1 & -1 & -2 \end{pmatrix}\begin{pmatrix} 1 & 1 & 3 \\ 1 & 1 & -3 \\ 3 & -3 & -3 \end{pmatrix}\begin{pmatrix} \sqrt{3} & \sqrt{2} & 1 \\ \sqrt{3} & -\sqrt{2} & -1 \\ 0 & \sqrt{2} & -2 \end{pmatrix} \tag{4}$$

$$= \frac{1}{6}\begin{pmatrix} \sqrt{3} & \sqrt{3} & 0 \\ \sqrt{2} & -\sqrt{2} & \sqrt{2} \\ 1 & -1 & -2 \end{pmatrix}\begin{pmatrix} 2\sqrt{3} & 3\sqrt{2} & -6 \\ 2\sqrt{3} & -3\sqrt{2} & 6 \\ 0 & 3\sqrt{2} & 12 \end{pmatrix} \tag{5}$$

$$= \frac{1}{6}\begin{pmatrix} 12 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & -36 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -6 \end{pmatrix} \tag{6}$$

- Which is diagonal, as required, and has elements with the eigenvalues of A $(2, 3, -6)$ QED.

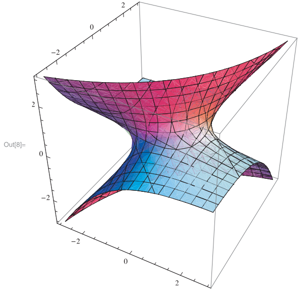
16.4 Example: changing the basis of a hyperboloid

An application of the diagonalization of a matrix.

- Take the hyperboloid $x^2 + y^2 - 3z^2 + 2xy + 6xz - 6yz = 4$
- In matrix form $\rightarrow \quad (X^T A X) = k$

$$(x, y, z) \begin{pmatrix} 1 & 1 & 3 \\ 1 & 1 & -3 \\ 3 & -3 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 4 \qquad (7)$$

(the same equation and matrix of the previous lectures)



Hyperboloid in the $|\mathbf{e}\rangle$ basis i.e. the (x, y, z) Cartesian frame

- Diagonalize A by taking eigenvalues/vectors etc

  $\rightarrow \; A' = S^{-1}AS$

  A' is then the diagonal matrix of eigenvalues of A

$$A' = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -6 \end{pmatrix} \tag{8}$$

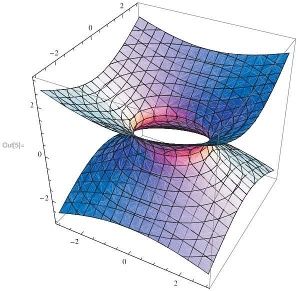
- In the $|\mathbf{e}'\rangle$ basis, the transformed frame, the hyperboloid is

$$(x', y', z') \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -6 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = 4 \tag{9}$$

$$\rightarrow \quad 2x'^2 + 3y'^2 - 6z'^2 = 4$$

- Hence the transformation $X^TAX \rightarrow X'^TA'X'$ transforms to an orthogonal basis $|\mathbf{e}'\rangle$ representing the hyperboloid axes; The axes of $|\mathbf{e}'\rangle$ w.r.t. $|\mathbf{e}\rangle$ are given by the eigenvectors of A

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix} \text{ and } \begin{pmatrix} \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{6}} \end{pmatrix} \tag{10}$$



Hyperboloid in (x', y', z') frame

- Form of equation of hyperboloid in the new basis $|\mathbf{e}'\rangle$ is
$$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} - \frac{z'^2}{c^2} = 4$$
where $a^2 = \frac{1}{2}$, $b^2 = \frac{1}{3}$ and $c^2 = \frac{1}{6}$
- The normalized eigenvectors of A give the direction of the $|\mathbf{e}'\rangle$ basis axes w.r.t. $|\mathbf{e}\rangle$.

FIN : END OF COURSE !!!