Classical Mechanics

LECTURE 15: MORE ON ANGULAR VARIABLES. CENTRAL FORCES.

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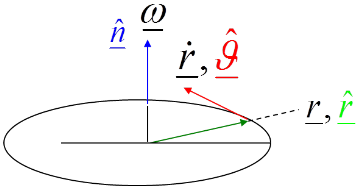
OUTLINE : 15. MORE ON ANGULAR VARIABLES. CENTRAL FORCES.

15.1 Angular acceleration $\underline{\alpha}$ for rotation in a circle

Angular velocity for rotation in a circle : $\underline{\dot{\mathbf{r}}} = \underline{\omega} \times \underline{\mathbf{r}}$

- $\underline{\omega} = \omega\, \underline{\hat{\mathbf{n}}} = \dot{\theta}\, \underline{\hat{\mathbf{n}}}$
- Angular acceleration:

$$\underline{\alpha} = \underline{\dot{\omega}}$$



Special case if $\underline{\alpha}$ is constant $\rightarrow$

- $\frac{d\omega}{dt} = \alpha \;\rightarrow\; \omega = \omega_0 + \alpha\, t$
- $\frac{d\theta}{dt} = \omega \;\rightarrow\; \theta = \theta_0 + \omega_0\, t + \frac{1}{2}\alpha\, t^2$

  Which should be recognisable equations !

Relationship between $\underline{\tau}$ and $\underline{\alpha}$ for rotation in a circle

$$\underline{\tau} = \frac{d}{dt}\underline{\mathbf{J}} = \mathrm{I}\underline{\alpha}$$

15.2 Angular motion : work and power

- Work linear motion :

  $dW = \underline{\mathbf{F}} \cdot d\underline{\mathbf{s}}$

  $\rightarrow \boxed{W = \int \underline{\mathbf{F}} \cdot d\underline{\mathbf{s}}}$

- Work angular motion :

  $\underline{\tau} = \underline{\mathbf{r}} \times \underline{\mathbf{F}}$

  $d\underline{\mathbf{s}} = d\underline{\theta} \times \underline{\mathbf{r}}$ ($d\underline{\theta}$ out of page)

  $dW = \underline{\mathbf{F}} \cdot d\underline{\mathbf{s}} = \underline{\mathbf{F}} \cdot (d\underline{\theta} \times \underline{\mathbf{r}})$

  $= (\underline{\mathbf{r}} \times \underline{\mathbf{F}}) \cdot d\underline{\theta}$

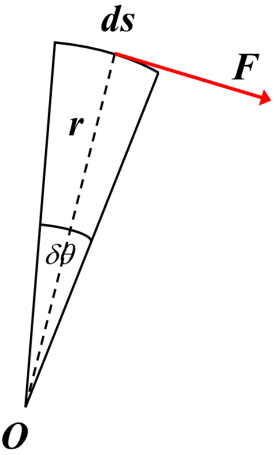
  (scalar triple product)

  $\boxed{W = \int \underline{\tau} d\underline{\theta} = \int \underline{\tau} \cdot \underline{\omega}\, dt}$

- Power :

  Linear motion : $P = \underline{\mathbf{F}} \cdot \underline{\mathbf{v}}$

  Rotational motion : $P = \underline{\tau} \cdot \underline{\omega}$



15.3 Correspondence between linear and angular quantities

Linear quantities are re-formulated in a rotating frame:

Linear/ translational quantities	Angular/ rotational quantities
Displacement, position: $\underline{\mathbf{r}}$ [m]	Angular displacement, angle: θ [rad]
Velocity: $\underline{\mathbf{v}}$ [m s^{-1}]	Angular velocity: $\underline{\omega}$ [rad s^{-1}]
Acceleration: $\underline{\mathbf{a}}$ [m s^{-2}]	Angular acceleration: $\underline{\alpha}$ [rad s^{-2}]
Mass m [kg]	Moment of inertia: I [kg m^2 rad^{-1}]
Momentum: $\underline{\mathbf{p}}$ [kg m s]$^{-1}$	Angular momentum: $\underline{\mathbf{J}}$ [kg m^2 s^{-1}]
Force $\underline{\mathbf{F}}$ [N = kg m s^{-2}]	Torque: $\underline{\tau}$ [kg m^2 s^{-2} rad^{-1}]
Weight F_g [N]	Moment [N m]
Work $dW = F \cdot dx$ [N m]	Work $W = \tau \cdot d\theta$ [N m]

15.3.1 Reformulation of Newton's laws for angular motion

1. In the absence of a net applied torque, the angular velocity remains unchanged.

2. Torque = [moment of inertia]$\times$ [angular acceleration]
   $\underline{\tau} = \mathtt{I}\underline{\alpha}$

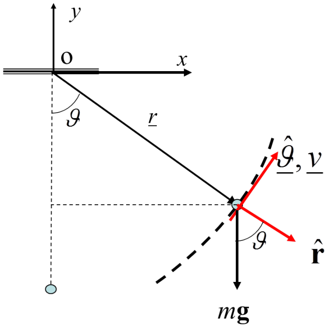
   This expression applies to rotation about a single principal axis, usually the axis of symmetry.
   (cf. $\underline{\mathbf{F}} = m\underline{\mathbf{a}}$). More on moment of inertia comes later.

3. For every applied torque, there is an equal and opposite reaction torque. (A result of Newton's 3rd law of linear motion.)

15.3.2 Example: the simple pendulum

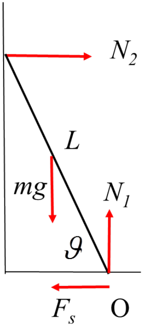
Derive the EOM of a simple pendulum using angular variables:

- $\underline{\tau} = \underline{\mathbf{r}} \times \underline{\mathbf{F}} = -mgr\sin\theta\,\underline{\hat{\mathbf{z}}}$
- $\underline{\mathbf{J}} = \underline{\mathbf{r}} \times m\underline{\mathbf{v}} = mrv\,\underline{\hat{\mathbf{z}}}$
- $v = r\dot{\theta} \quad \rightarrow \quad \dot{v} = r\ddot{\theta}$
- $\frac{d\underline{\mathbf{J}}}{dt} = mr\dot{v}\,\underline{\hat{\mathbf{z}}} = (mr^2\ddot{\theta})\,\underline{\hat{\mathbf{z}}}$
  (since $\underline{\hat{\mathbf{z}}}$ is a constant vector)
- $\frac{d\underline{\mathbf{J}}}{dt} = \underline{\tau} \quad \rightarrow \quad mr^2\ddot{\theta} = -mgr\sin\theta$
- $\ddot{\theta} + \frac{g}{r}\sin\theta = 0$



15.4 Moments of forces

Simple example : ladder against a wall



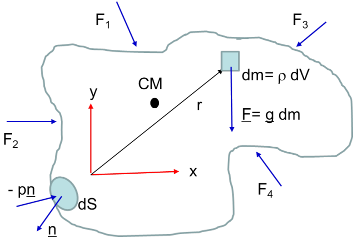
- If no slipping, torques (moments) must balance
- About any point:

  $\sum_{i=1}^{n} \underline{\mathbf{r}}_i \times \underline{\mathbf{F}}_i = \underline{\tau}_{tot} = 0$
- Moments about O

  $mg\frac{L}{2}\cos\theta = N_2 L \sin\theta$
- Also balance of forces in equilibrium

  $mg = N_1 \; and \; F_s = \mu N_1 = N_2$

General case: body subject to gravity.
Total moment :



- $\underline{\mathbf{M}} = \int_V \underline{\mathbf{r}} \times \underline{\mathbf{g}}\, \rho\, dV$ mass term

  $+\sum_{i=1}^{n} \underline{\mathbf{r}}_i \times \underline{\mathbf{F}}_i$ external forces

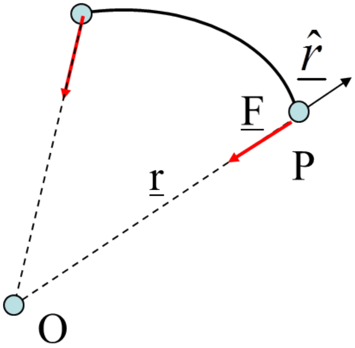
  $-\int_S \underline{\mathbf{r}} \times (p\underline{\mathbf{n}}\, dS)$ surface pressure

15.5 Central forces

- Central force: $\underline{\mathbf{F}}$ acts towards origin (line joining O and P) always.
- $\underline{\mathbf{F}} = f(r)\,\hat{\underline{\mathbf{r}}}$ only
- Examples:

  Gravitational force $\underline{\mathbf{F}} = -\frac{GmM}{r^2}\hat{\underline{\mathbf{r}}}$

  Electrostatic force $\underline{\mathbf{F}} = \frac{q_1 q_2}{4\pi\epsilon_0 r^2}\hat{\underline{\mathbf{r}}}$



15.5.1 A central force is conservative

A force $\underline{\mathbf{F}}$ is conservative if it meets 3 *equivalent* conditions:

1. The curl of $\underline{\mathbf{F}}$ is zero : $\nabla \times \underline{\mathbf{F}} = 0$
2. Work over closed path $W \equiv \oint_C \underline{\mathbf{F}} \cdot \mathrm{d}\underline{\mathbf{r}} = 0$, independent of path
3. $\underline{\mathbf{F}}$ can be written in terms of scalar potential $\underline{\mathbf{F}} = -\nabla U$

- Equivalence of 1 & 2 from Stokes' theorem
  $\int_S (\nabla \times \underline{\mathbf{F}}) \cdot \mathrm{d}\underline{\mathbf{a}} = \oint_C \underline{\mathbf{F}} \cdot \mathrm{d}\underline{\mathbf{r}} = 0$
- Equivalence of 1 & 3 from vector calculus identity :
  $\nabla \times (\nabla U) = 0$

For a *central* potential, take the grad of $U(r)$:

- In cartesians $\nabla U(r) = \frac{\partial U(\sqrt{x^2+y^2+z^2})}{\partial x} \hat{\underline{\mathbf{x}}} + \ldots$ ($\hat{\underline{\mathbf{y}}}$ and $\hat{\underline{\mathbf{z}}}$ terms)
- Chain rule $\frac{\partial U}{\partial x} = \frac{\partial U}{\partial r}\frac{\partial r}{\partial x}$: $\nabla U(r) = \frac{x}{\sqrt{x^2+y^2+z^2}} \frac{\partial U(r)}{\partial r} \hat{\underline{\mathbf{x}}} + \ldots$
- Since $\frac{x\hat{\underline{\mathbf{x}}}+y\hat{\underline{\mathbf{y}}}+z\hat{\underline{\mathbf{z}}}}{\sqrt{x^2+y^2+z^2}} = \hat{\underline{\mathbf{r}}} \rightarrow -\nabla U(r) = -\frac{\partial U(r)}{\partial r}\hat{\underline{\mathbf{r}}} \equiv f(r)\hat{\underline{\mathbf{r}}} = \underline{\mathbf{F}}(\underline{\mathbf{r}})$

The grad of the scalar potential has only one non-vanishing component which is along $\hat{\underline{\mathbf{r}}}$ ($\rightarrow$ central force).
Hence condition (3) is satisfied $\rightarrow$ central force is conservative force.