Classical Mechanics

LECTURE 14:

INTRODUCTION TO TORQUE AND ANGULAR MOMENTUM

Prof. N. Harnew

University of Oxford

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OUTLINE : 14. INTRODUCTION TO TORQUE AND ANGULAR MOMENTUM

14.1 Differentiation of vectors wrt time

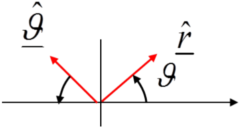
Vectors follow the rules of differentiation:

- $\frac{d}{dt}\underline{\mathbf{a}} = \frac{da_x}{dt}\underline{\mathbf{i}} + \frac{da_y}{dt}\underline{\mathbf{j}} + \frac{da_z}{dt}\underline{\mathbf{k}} = \dot{a}_x\underline{\mathbf{i}} + \dot{a}_y\underline{\mathbf{j}} + \dot{a}_z\underline{\mathbf{k}}$
- $\frac{d}{dt}(\underline{\mathbf{a}} + \underline{\mathbf{b}}) = \frac{d\underline{\mathbf{a}}}{dt} + \frac{d\underline{\mathbf{b}}}{dt} = \dot{\underline{\mathbf{a}}} + \dot{\underline{\mathbf{b}}}$
- $\frac{d}{dt}(c\,\underline{\mathbf{a}}) = \frac{d\,c}{dt}\,\underline{\mathbf{a}} + c\,\frac{d\underline{\mathbf{a}}}{dt} = \dot{c}\,\underline{\mathbf{a}} + c\,\dot{\underline{\mathbf{a}}}$
- $\frac{d}{dt}(\underline{\mathbf{a}}.\underline{\mathbf{b}}) = \frac{d\underline{\mathbf{a}}}{dt}.\underline{\mathbf{b}} + \underline{\mathbf{a}}.\frac{d\underline{\mathbf{b}}}{dt} = \dot{\underline{\mathbf{a}}}.\underline{\mathbf{b}} + \underline{\mathbf{a}}.\dot{\underline{\mathbf{b}}}$
- $\frac{d}{dt}(\underline{\mathbf{a}} \times \underline{\mathbf{b}}) = \frac{d\underline{\mathbf{a}}}{dt} \times \underline{\mathbf{b}} + \underline{\mathbf{a}} \times \frac{d\underline{\mathbf{b}}}{dt} = \dot{\underline{\mathbf{a}}} \times \underline{\mathbf{b}} + \underline{\mathbf{a}} \times \dot{\underline{\mathbf{b}}}$ (order is impt.)

Orthogonality of differentiated unit vectors

- $\frac{d}{dt}(\underline{\hat{\mathbf{r}}}.\underline{\hat{\mathbf{r}}}) = 2\underline{\hat{\mathbf{r}}}.\frac{d\underline{\hat{\mathbf{r}}}}{dt} = 0$ (since $\underline{\hat{\mathbf{r}}}.\underline{\hat{\mathbf{r}}} = 1$)

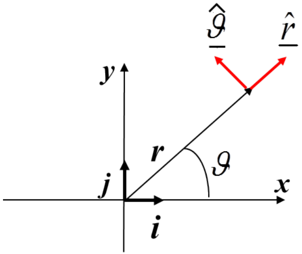
  Therefore $\frac{d\underline{\hat{\mathbf{r}}}}{dt} \perp \underline{\hat{\mathbf{r}}} \rightarrow \frac{d\underline{\hat{\mathbf{r}}}}{dt} \propto \underline{\hat{\theta}}$

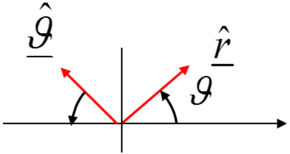


Derivative of any unit vector gives a vector perpendicular to it.

14.1.1 The position vector in polar coordinates

- $\underline{\mathbf{r}} = r_0(\underline{\mathbf{i}} \cos\theta + \underline{\mathbf{j}} \sin\theta)$
  $\underline{\hat{\mathbf{r}}} = (\underline{\mathbf{i}} \cos\theta + \underline{\mathbf{j}} \sin\theta)$ is a unit vector in the direction of $\underline{\mathbf{r}}$
- $\frac{d\underline{\hat{\mathbf{r}}}}{dt} = \left[-\underline{\mathbf{i}} \sin\theta + \underline{\mathbf{j}} \cos\theta\right] \dot{\theta}$
- $\underline{\hat{\theta}} = (-\underline{\mathbf{i}} \sin\theta + \underline{\mathbf{j}} \cos\theta)$ is a unit vector perpendicular to $\underline{\mathbf{r}}$
- $\dot{\underline{\hat{\mathbf{r}}}} = \dot{\theta}\,\underline{\hat{\theta}}$ also $\dot{\underline{\hat{\theta}}} = -\dot{\theta}\,\underline{\hat{\mathbf{r}}}$



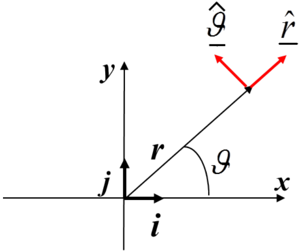


14.1.2 The velocity vector in polar coordinates

- $\underline{\mathbf{r}} = r\,\underline{\hat{\mathbf{r}}}$

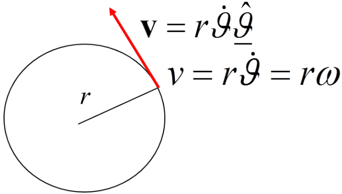
  $\underline{\mathbf{v}} = \underline{\dot{\mathbf{r}}} = \dot{r}\,\underline{\hat{\mathbf{r}}} + r\,\underline{\dot{\hat{\mathbf{r}}}}$
- From before $\underline{\dot{\hat{\mathbf{r}}}} = \dot{\theta}\,\underline{\hat{\theta}}$

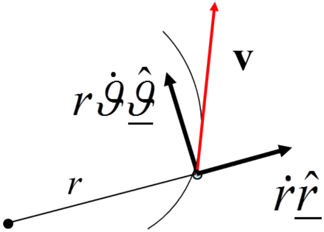
General case: $\underline{\mathbf{v}} = \underline{\dot{\mathbf{r}}} = \dot{r}\,\underline{\hat{\mathbf{r}}} + r\,\dot{\theta}\,\underline{\hat{\theta}}$



For circular motion:

- Since $\dot{r} = 0$
- $\underline{\mathbf{v}} = r\,\dot{\theta}\,\underline{\hat{\theta}} = r\,\omega\,\underline{\hat{\theta}}$





14.1.3 The acceleration vector in polar coordinates

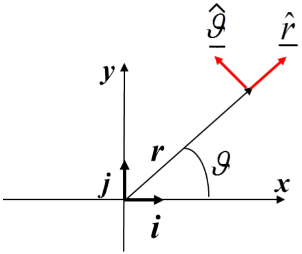
- From before $\underline{\mathbf{v}} = \underline{\dot{\mathbf{r}}} = \dot{r}\,\underline{\hat{\mathbf{r}}} + r\,\dot{\theta}\,\underline{\hat{\theta}}$
- $\underline{\mathbf{a}} = \underline{\dot{\mathbf{v}}} = \underline{\ddot{\mathbf{r}}}$

  $\frac{d}{dt}(\dot{r}\,\underline{\hat{\mathbf{r}}}) = \ddot{r}\underline{\hat{\mathbf{r}}} + \dot{r}\dot{\theta}\underline{\hat{\theta}}$ (since $\underline{\dot{\hat{\mathbf{r}}}} = \dot{\theta}\,\underline{\hat{\theta}}$)

  $\frac{d}{dt}(r\,\dot{\theta}\,\underline{\hat{\theta}}) = r\,\dot{\theta}\,\underline{\dot{\hat{\theta}}} + r\,\ddot{\theta}\,\underline{\hat{\theta}} + \dot{r}\,\dot{\theta}\,\underline{\hat{\theta}}$

  $= -r\,\dot{\theta}^2\,\underline{\hat{\mathbf{r}}} + r\,\ddot{\theta}\,\underline{\hat{\theta}} + \dot{r}\,\dot{\theta}\,\underline{\hat{\theta}}$

  (since $\underline{\dot{\hat{\theta}}} = -\dot{\theta}\,\underline{\hat{\mathbf{r}}}$)



General case :

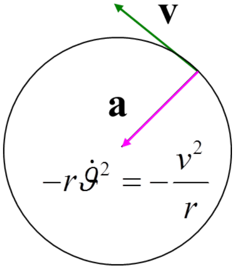
$$\underline{\mathbf{a}} = \underline{\ddot{\mathbf{r}}} = (\ddot{r} - r\,\dot{\theta}^2)\,\underline{\hat{\mathbf{r}}} + (2\dot{r}\,\dot{\theta} + r\,\ddot{\theta})\,\underline{\hat{\theta}}$$

For circular motion:

- Since scalars $\ddot{r} = \dot{r} = \ddot{\theta} = 0$

  (no change in magnitudes of radius or azimuthal acceleration)

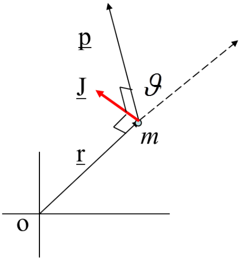
  $\underline{\mathbf{a}} = -r\,\dot{\theta}^2\,\underline{\hat{\mathbf{r}}} = -\omega^2 r\,\underline{\hat{\mathbf{r}}} = -\frac{v^2}{r}\,\underline{\hat{\mathbf{r}}}$



14.2 Angular momentum and torque

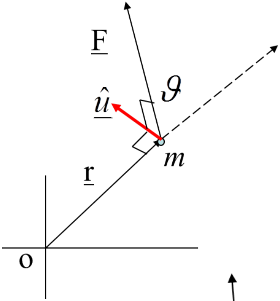
- The definition of *angular momentum* (or the *moment of momentum*) $\underline{\mathbf{J}}$ for a single particle : $\underline{\mathbf{J}} = \underline{\mathbf{r}} \times \underline{\mathbf{p}}$

  $\underline{\mathbf{r}}$ is the displacement vector from the origin and $\underline{\mathbf{p}}$ the momentum



- The direction of the angular momentum gives the direction perpendicular to the plane of motion
  - Differentiate: $\frac{d\underline{\mathbf{J}}}{dt} = \underline{\mathbf{r}} \times \frac{d\underline{\mathbf{p}}}{dt} + \frac{d\underline{\mathbf{r}}}{dt} \times \underline{\mathbf{p}}$
  - Definitions of force and velocity: $\underline{\mathbf{F}} = \frac{d\underline{\mathbf{p}}}{dt}$ and $\underline{\mathbf{v}} = \frac{d\underline{\mathbf{r}}}{dt}$
  - $\frac{d\underline{\mathbf{J}}}{dt} = \underline{\mathbf{r}} \times \underline{\mathbf{F}} + \underline{\mathbf{v}} \times \underline{\mathbf{p}}$ $\leftarrow$ this term$= m\underline{\mathbf{v}} \times \underline{\mathbf{v}} = 0$
  - Define torque $\underline{\tau} = \underline{\mathbf{r}} \times \underline{\mathbf{F}} = \frac{d\underline{\mathbf{J}}}{dt}$ (cf. Linear motion $\underline{\mathbf{F}} = \frac{d\underline{\mathbf{p}}}{dt}$)
  - For multiple forces : $\frac{d\underline{\mathbf{J}}}{dt} = \sum_{i=1}^{n} \underline{\mathbf{r}}_i \times \underline{\mathbf{F}}_i = \underline{\tau}_{tot}$

Torque depends on the origin



- Torque wrt origin O

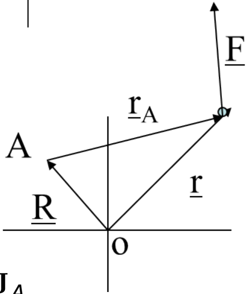
  $\underline{\tau}_o = \underline{\mathbf{r}} \times \underline{\mathbf{F}}$

- Torque wrt point A

  $\underline{\tau}_A = \underline{\mathbf{r}}_A \times \underline{\mathbf{F}} = \underline{\mathbf{r}} \times \underline{\mathbf{F}} - \underline{\mathbf{R}} \times \underline{\mathbf{F}}$

  $= \underline{\tau}_0 - \underline{\mathbf{R}} \times \underline{\mathbf{F}}$

- Hence in general $\underline{\tau}_o \neq \underline{\tau}_A$

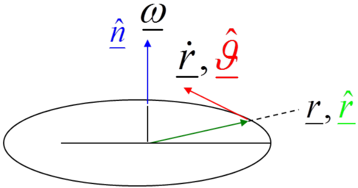


Same applies to angular momentum : $\underline{\mathbf{J}}_o \neq \underline{\mathbf{J}}_A$

14.3 Angular velocity $\underline{\omega}$ for rotation in a circle

- Definition of angular velocity :

$$\underline{\dot{\mathbf{r}}} = \underline{\omega} \times \underline{\mathbf{r}}$$



- Note that $\underline{\dot{\mathbf{r}}}$ is *always* $\perp$ $\underline{\mathbf{r}}$, so $\underline{\omega}$ is defined for circular motion
- Define $\underline{\hat{\mathbf{n}}}$ such that $\underline{\hat{\theta}} = \underline{\hat{\mathbf{n}}} \times \underline{\hat{\mathbf{r}}}$
- Recall $\underline{\mathbf{v}} = \underline{\dot{\mathbf{r}}} = \dot{r}\underline{\mathbf{r}} + r\dot{\theta}\underline{\hat{\theta}}$
- For circular motion $\dot{r} = 0$; $\dot{\theta} = \omega$

$\rightarrow$ $\underline{\dot{\mathbf{r}}} = \underline{\omega} \times \underline{\mathbf{r}} = (\omega\, \underline{\hat{\mathbf{n}}}) \times (r\, \underline{\hat{\mathbf{r}}}) = r\omega\, \underline{\hat{\theta}}$

Relationship between $\underline{\mathbf{J}}$ and $\underline{\omega}$

- $\underline{\mathbf{J}} = \underline{\mathbf{r}} \times \underline{\mathbf{p}} = m\,\underline{\mathbf{r}} \times \underline{\dot{\mathbf{r}}} = m\,\underline{\mathbf{r}} \times (\underline{\omega} \times \underline{\mathbf{r}})$
- Recall vector identity $\underline{\mathbf{a}} \times (\underline{\mathbf{b}} \times \underline{\mathbf{c}}) = (\underline{\mathbf{a}}.\underline{\mathbf{c}})\,\underline{\mathbf{b}} - (\underline{\mathbf{a}}.\underline{\mathbf{b}})\,\underline{\mathbf{c}}$
- $\underline{\mathbf{J}} = m\,r^2\,\underline{\omega} - m\,(\underline{\mathbf{r}} \cdot \underline{\omega})\,\underline{\mathbf{r}}$
- $\underline{\mathbf{r}} \cdot \underline{\omega} = 0$ since the circular rotation is in a plane
- Hence $\underline{\mathbf{J}} = \mathtt{I}\underline{\omega}$ where $\mathtt{I} = mr^2$; (generally $\mathtt{I} = \sum_i [m_i r_i^2]$)