# *LECTURE 14:*

# *EXAMPLES OF CHANGE OF BASIS AND MATRIX TRANSFORMATIONS. QUADRATIC FORMS.*

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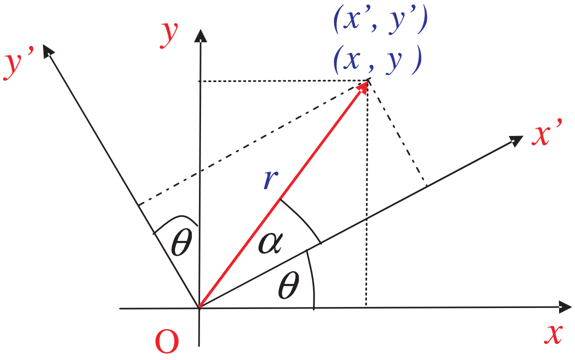
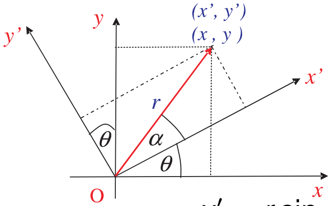
# *Outline: 14. EXAMPLES OF CHANGE OF BASIS AND MATRIX TRANSFORMATIONS. QUADRATIC FORMS.*

# *14.1 Examples of change of basis*

## *14.1.1 Representation of a 2D vector in a rotated coordinate frame*




- Transformation of vector $\underline{\mathbf{r}}$ from Cartesian axes $(x, y)$ into frame $(x', y')$, rotated by angle $\theta$




$$x' = r\cos\alpha$$
$$x = r\cos(\theta+\alpha)$$
$$\rightarrow \quad x' = \frac{x\cos\alpha}{\cos(\theta+\alpha)}$$
$$x\cos\alpha = x'\cos\theta\cos\alpha - x'\sin\theta\sin\alpha$$
Since $\quad x'\sin\alpha = y'\cos\alpha$

$$x = x'\cos\theta - y'\sin\theta$$

$$y' = r\sin\alpha$$
$$y = r\sin(\theta+\alpha)$$
$$\rightarrow \quad y' = \frac{y\sin\alpha}{\sin(\theta+\alpha)}$$
$$y\sin\alpha = y'\sin\theta\cos\alpha + y'\cos\theta\sin\alpha$$
Since $\quad y'\cos\alpha = x'\sin\alpha$

$$y = x'\sin\theta + y'\cos\theta$$

- Coordinate transformation:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} \qquad (1)$$

- Take the inverse:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \qquad (2)$$

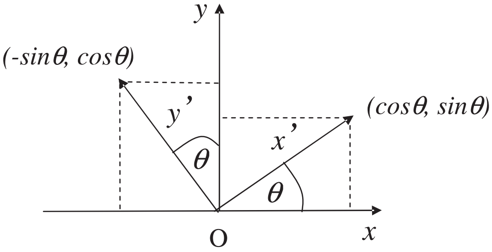
These equations relate the coordinates of $\underline{\mathbf{r}}$ measured in the $(x, y)$ frame with those measured in the rotated $(x', y')$ frame

# 14.1.2 Rotation of a coordinate system in 2D

- Start from the familiar orthonormal basis

$$|\mathbf{e_1}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \;\; (\equiv \underline{\hat{\mathbf{x}}}), \qquad |\mathbf{e_2}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \;\; (\equiv \underline{\hat{\mathbf{y}}}) \qquad (3)$$

- Transform the basis via a rotation through an angle $\theta$




$$\text{New basis}: \;\; |\mathbf{e}_1'\rangle = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \;\; (\equiv \underline{\hat{\mathbf{x}}'}), \quad |\mathbf{e}_2'\rangle = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \;\; (\equiv \underline{\hat{\mathbf{y}}'}) \qquad (4)$$

- The transformation matrix $S$ is determined from $|\mathbf{e_i'}\rangle = S|\mathbf{e_i}\rangle$

$$\begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \Rightarrow \begin{array}{l} S_{11} = \cos\theta \\ S_{21} = \sin\theta \end{array} \tag{5}$$

$$\begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \Rightarrow \begin{array}{l} S_{12} = -\sin\theta \\ S_{22} = \cos\theta \end{array} \tag{6}$$

$$\text{Hence} \quad S(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{7}$$

  As expected, the basis transformation matrix $|\mathbf{e}\rangle \to |\mathbf{e}'\rangle$ is the inverse of the transformation $(x, y) \to (x', y')$ of the components derived in the previous sub-section.

- The inverse transformation matrix rotates backwards

$$S^{-1}(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \equiv S(-\theta) \tag{8}$$

- It is easy to show via substitution that two successive rotations

$$S(\theta)S(\alpha) = S(\alpha)S(\theta) = S(\theta + \alpha)$$

## *14.2 Rotation of a vector in fixed 3D coord. system*

- In 3D, we can rotate a vector $\underline{\mathbf{r}}$ about any one of the three axes

$$\underline{\mathbf{r}}' = R(\theta)\,\underline{\mathbf{r}}$$

A rotation about the $z$ axis is given by

$$R_z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{9}$$

Note that rotations of a vector in a fixed coordinate system transform in the same way as rotations of the base vectors (see previous section).

- For rotations about the $x$ and $y$ axes

$$R_x(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}, \quad R_y(\gamma) = \begin{pmatrix} \cos\gamma & 0 & \sin\gamma \\ 0 & 1 & 0 \\ -\sin\gamma & 0 & \cos\gamma \end{pmatrix} \tag{10}$$

- But note that now for successive rotations:

$$R_z(\theta)R_x(\alpha) \neq R_x(\alpha)R_z(\theta)$$

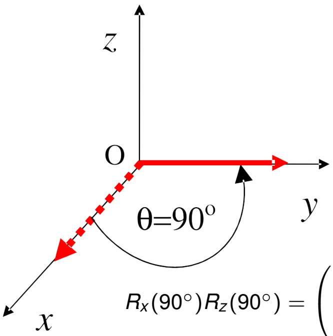
# *14.2.1 Example 1*

Rotate the unit vector $(1,0,0)$ by $90^\circ$ about the $z$-axis

$$R_z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$\rightarrow$ Which is a unit vector along the $y$-axis (as expected).




- Now make a $2^{nd}$ rotation of $90^\circ$ about the $x$-axis:

$$R_x(90^\circ)R_z(90^\circ) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

So ... by now the procedure of matrix multiplication should be clear: the exact form of the row/column multiplication is necessary to make a *linear transformation* between two bases. It is also the required form for *rotations of vectors* in their associated vector space(s).

## *14.2.2 Example 2*

- Rotate the vector $\underline{\mathbf{r}} = (1, 2, 3)$ by $30^\circ$ about the $y$ axis.
  $\sin 30^\circ = 1/2; \quad \cos 30^\circ = \sqrt{3}/2$
- The rotation matrix is

$$R_y(\gamma) = \begin{pmatrix} \cos\gamma & 0 & \sin\gamma \\ 0 & 1 & 0 \\ -\sin\gamma & 0 & \cos\gamma \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ -1/2 & 0 & \sqrt{3}/2 \end{pmatrix} \tag{11}$$

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ -1/2 & 0 & \sqrt{3}/2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 + 3/2 \\ 2 \\ -1/2 + 3\sqrt{3}/2 \end{pmatrix} \tag{12}$$

- As a check - rotate back $\rightarrow$ use inverse matrix

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 & 0 & -1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & \sqrt{3}/2 \end{pmatrix} \begin{pmatrix} \sqrt{3}/2 + 3/2 \\ 2 \\ -1/2 + 3\sqrt{3}/2 \end{pmatrix} \tag{13}$$

$$= \begin{pmatrix} 3/4 + 3\sqrt{3}/4 + 1/4 - 3\sqrt{3}/4 \\ 2 \\ \sqrt{3}/4 + 3/4 - \sqrt{3}/4 + 9/4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \text{ as required.} \tag{14}$$

## 14.3 MATRICES AND QUADRATIC FORMS

*Best illustrated by a few examples.*

### 14.3.1 Example 1: a $2 \times 2$ quadratic form

- Represent equation $x^2 + y^2 = 1$ in matrix form $X^T AX = 1$
- Matrix $A$ is a transformation matrix which represents the conic form of the equation.

$$x^2 + y^2 = (x, y) \begin{pmatrix} x \\ y \end{pmatrix} = (x, y) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 1 \tag{15}$$

## *14.3.2 Example 2: another $2 \times 2$ quadratic form*

- Matrix representation:

$$(x, y) \begin{pmatrix} 5 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 5 \tag{16}$$

- Write in equation form:

$$= \; (x, y) \begin{pmatrix} 5x + 2y \\ 2x + 3y \end{pmatrix} \tag{17}$$

$$= 5x^2 + (2xy + 2xy) + 3y^2 \; = \; 5x^2 + 4xy + 3y^2$$

$$\rightarrow \;\; 5x^2 + 4xy + 3y^2 = 5$$

## 14.3.3 Example 3: a 3 × 3 quadratic form

- Represent $x^2 + y^2 - 3z^2 + 2xy + 6xz - 6yz = 4$ in the matrix form ($X^T AX$).
- Write

$$(x, y, z)\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = (x, y, z)\begin{pmatrix} ax + by + cz \\ dx + ey + fz \\ gx + hy + iz \end{pmatrix} \quad (18)$$

$$= ax^2 + bxy + cxz + dxy + ey^2 + fyz + gxz + hyz + iz^2$$

- Comparing terms:
  $[x^2] \rightarrow a = 1; \quad [y^2] \rightarrow e = 1; \quad [z^2] \rightarrow i = -3$
  $[xy] \rightarrow b + d = 2; \quad [xz] \rightarrow c + g = 6; \quad [yz] \rightarrow f + h = -6$
  (underconstrained)
- In echelon form:
  set b = c = f = 0

$$X^T\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 6 & -6 & -3 \end{pmatrix}X = 4 \quad (19)$$

- In symmetrical form:
  set b = d, c = g, f = h

$$X^T\begin{pmatrix} 1 & 1 & 3 \\ 1 & 1 & -3 \\ 3 & -3 & -3 \end{pmatrix}X = 4 \quad (20)$$