Classical Mechanics

LECTURE 13:

MOTION OF CHARGE IN E & B FIELDS

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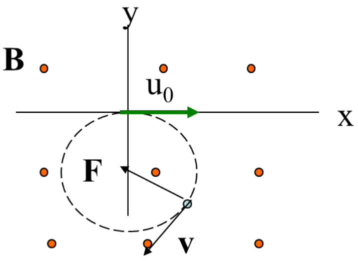
OUTLINE : 13. MOTION OF CHARGE IN E & B FIELDS

13.1 Magnetic Force on a Charged Particle

$$\underline{\mathbf{F}} = q\,\underline{\mathbf{v}} \times \underline{\mathbf{B}}$$

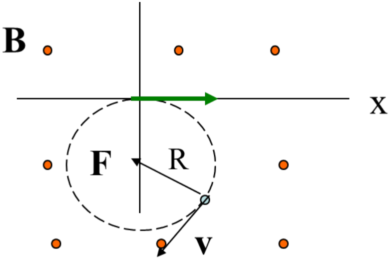
- $\underline{\mathbf{F}}$ is the magnetic force
- q is the charge
- $\underline{\mathbf{v}}$ is the velocity of the moving charge
- $\underline{\mathbf{B}}$ is the magnetic field



Because the magnetic force is perpendicular to the displacement ($dW = \underline{\mathbf{F}}.\underline{\mathbf{dx}}$), the force does no work on the particle

- Kinetic energy does not change
- Speed does not change
- Only direction changes
- Particle moves in a circle if $v \perp B$

$\underline{\mathbf{F}} = q\,\underline{\mathbf{v}} \times \underline{\mathbf{B}}$



Newton's second law in components

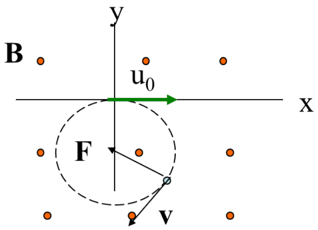
- $m\ddot{x} = F_x$
- $m\ddot{y} = F_y$
- $m\ddot{z} = F_z$

Simple case: $\underline{\mathbf{v}}$ perpendicular to $\underline{\mathbf{B}}$

- Magnetic force= centripetal force
- $F = qvB = \frac{mv^2}{R}$ (magnitudes)
  where R is the radius of curvature
- $R = \frac{mv}{qB} = \frac{p}{qB}$
  p is the particle momentum

13.2 B Field only, $\underline{\mathbf{v}} \perp \underline{\mathbf{B}}$

$$m\ddot{\underline{\mathbf{r}}} = q\underline{\mathbf{v}} \times \underline{\mathbf{B}}$$



- $m\ddot{x} = F_x,\ m\ddot{y} = F_y,\ m\ddot{z} = F_z$
- $\underline{\mathbf{B}}$−field only $\rightarrow$ $\underline{\mathbf{B}} = B_z\underline{\mathbf{k}}$
- $\underline{\mathbf{v}} = (\dot{x}, \dot{y}, 0)$

$$\underline{\mathbf{v}} \times \underline{\mathbf{B}} = \begin{vmatrix} \underline{\mathbf{i}} & \underline{\mathbf{j}} & \underline{\mathbf{k}} \\ \dot{x} & \dot{y} & \dot{z} \\ 0 & 0 & B_z \end{vmatrix} = \dot{y}B_z\underline{\mathbf{i}} - \dot{x}B_z\underline{\mathbf{j}}$$

- $m\ddot{x} = qB_z\dot{y}; \qquad m\ddot{y} = -qB_z\dot{x}$
- $\ddot{x} = \omega\dot{y};\ \ \ddot{y} = -\omega\dot{x}$ where $\omega = \frac{qB_z}{m}$
- $\dot{y} = -\omega x + c;\ \ (\dot{y} = 0 \text{ at } t = 0 \rightarrow c = 0)$
- $\ddot{x} + \omega^2 x = 0 \ \rightarrow\ x = A_1 \cos\omega t + A_2 \sin\omega t$
  - At $t = 0,\ x = 0 \ \rightarrow\ A_1 = 0 \ \rightarrow\ x = A_2 \sin\omega t$
  - $\dot{x} = A_2\,\omega \cos\omega t$; at $t = 0,\ \dot{x} = u_0 \ \rightarrow\ A_2 = \frac{u_0}{\omega}$

- At $t = 0$:
- $\underline{\mathbf{r}}_0 = (0, 0, 0)$
- $\underline{\mathbf{v}}_0 = (u_0, 0, 0)$

B Field only, continued

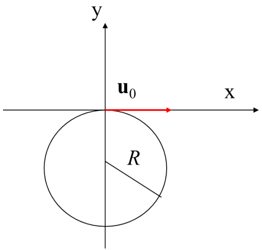
$$x = \frac{u_0}{\omega}\sin\omega t$$

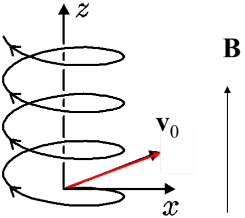
- From before :
  $\dot{y} = -\omega x = -u_0 \sin\omega t$
- $y = \frac{u_0}{\omega}\cos\omega t + c'$
- At $t = 0,\ y = 0 \ \rightarrow\ c' = -\frac{u_0}{\omega}$

$$y = \frac{u_0}{\omega}(\cos\omega t - 1)$$

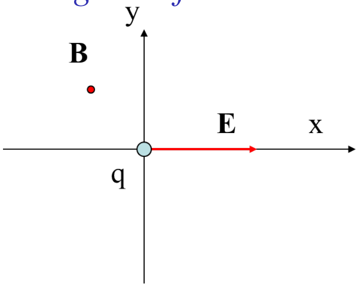
- $x = R\sin\omega t \ \ (R = \frac{u_0}{\omega} = \frac{u_0 m}{qB_z})$
- $y + R = R\cos\omega t$
  $\rightarrow$ Circle $x^2 + (y+R)^2 = R^2$
- If at $t = 0,\ \underline{\mathbf{v}}_0 = (u_0, 0, w_0)$
  The particle will spiral in circles about the z-direction:

$$z = w_0 t;\ \ x^2 + (y+R)^2 = R^2$$





13.3 Motion under electric and magnetic fields



$$m\ddot{\underline{\mathbf{r}}} = q\underline{\mathbf{v}} \times \underline{\mathbf{B}} + q\underline{\mathbf{E}}$$

- $m\ddot{x} = F_x, m\ddot{y} = F_y, m\ddot{z} = F_z$
- $\underline{\mathbf{B}}$–field→ $\underline{\mathbf{B}} = B_z\underline{\mathbf{k}}$
- $\underline{\mathbf{E}}$–field→ $\underline{\mathbf{E}} = E_x\underline{\mathbf{i}}$

$$\underline{\mathbf{v}} \times \underline{\mathbf{B}} = \begin{vmatrix} \underline{\mathbf{i}} & \underline{\mathbf{j}} & \underline{\mathbf{k}} \\ \dot{x} & \dot{y} & \dot{z} \\ 0 & 0 & B_z \end{vmatrix} = \dot{y}B_z\underline{\mathbf{i}} - \dot{x}B_z\underline{\mathbf{j}}$$

- $m\ddot{x} = qB_z\dot{y} + qE_x;$ $\quad m\ddot{y} = -qB_z\dot{x}$
- $\ddot{x} = \omega\dot{y} + \frac{qE_x}{m}$ where $\omega = \frac{qB_z}{m}$
- $\dot{y} = -\omega x$ (as before $\dot{y} = 0$ at $t = 0$)
- $\ddot{x} + \omega^2 x = \frac{qE_x}{m}$: solution $x = x_1 + x_2$
  - Complementary function : $x_1 = A_1 \cos\omega t + A_2 \sin\omega t$
  - Particular integral : $x_2 = \frac{qE_x}{m\omega^2}$

- At $t = 0$:
- $\underline{\mathbf{r}}_0 = (0, 0, 0)$
- $\underline{\mathbf{v}}_0 = (0, 0, 0)$

Electric and magnetic fields, continued

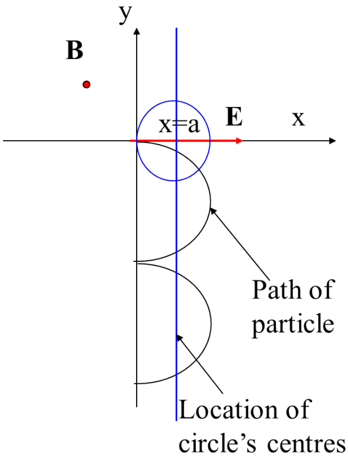
- $x = A_1 \cos\omega t + A_2 \sin\omega t + \frac{qE_x}{m\omega^2}$

  Define $a = \frac{qE_x}{m\omega^2}$
- $t = 0,\ x = 0 \ \rightarrow\ A_1 + a = 0$
- $t = 0,\ \dot{x} = 0 \ \rightarrow\ A_2 = 0$

  $$x = a\,(1 - \cos\omega t)$$
- From before :

  $\dot{y} = -\omega x = -a\omega\,(1 - \cos\omega t)$
- $y = a\,(\sin\omega t - \omega t) + c$
- At $t = 0,\ y = 0 \ \rightarrow\ c = 0$

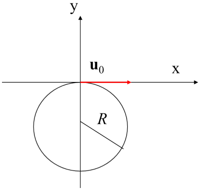
  $$y = a\,(\sin\omega t - \omega t)$$
- $-a\cos\omega t = x - a$
- $a\sin\omega t = y + a\omega t$



$\rightarrow$ A circle rolling down the $-y$ axis : $(x - a)^2 + (y + a\omega t)^2 = a^2$

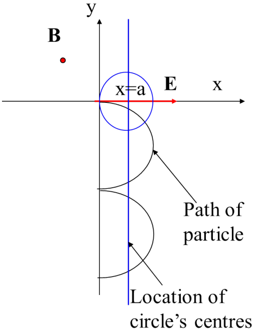
13.4 Kinetic energy in E & B fields

13.4.1 B only, $v \perp B$



- $x = \frac{u_0}{\omega} \sin \omega t \ \rightarrow \ \dot{x} = u_0 \cos \omega t$
- $y = \frac{u_0}{\omega} (\cos \omega t - 1)$
  $\rightarrow \ \dot{y} = -u_0 \sin \omega t$
- $T = \frac{1}{2} m [u_0 \cos \omega t]^2 + \frac{1}{2} m [-u_0 \sin \omega t]^2$
  $= \frac{1}{2} m u_0^2 \ \rightarrow$ No energy change

13.4.2 B and E



- $x = a (1 - \cos \omega t) \ \rightarrow \ \dot{x} = a \omega \sin \omega t$
- $y = a (\sin \omega t - \omega t)$
  $\rightarrow \ \dot{y} = a \omega (\cos \omega t - 1)$
- $T = \frac{1}{2} m (a \omega)^2 (1 + 1 - 2 \cos \omega t)$
  $= \frac{1}{2} m (2 a \omega)^2 \sin^2 \frac{\omega t}{2} \ \rightarrow$ W.D. by $\underline{\mathbf{E}}$ field

13.5 Cyclotron motion (E & B fields)

- Let the electric field vary with time as:

$$\underline{\mathbf{E}} = E_0 \begin{pmatrix} \cos\omega t \\ -\sin\omega t \\ 0 \end{pmatrix}, \quad \underline{\mathbf{B}} = B_z \underline{\mathbf{k}}$$

- Can show by direct substitution

$$x(t) = R\left[\omega\, t \sin\omega t + \sin\omega t - 1\right]$$

$$y(t) = R\left[\omega\, t \cos\omega t - \sin\omega t\right]$$

is a solution of the EOM

where $R = \frac{qE_0}{m\omega^2}$ and $\omega = \frac{qB_z}{m}$

ω is the Cyclotron frequency

- $(x+R)^2 + y^2 = R^2\left[(\omega t)^2 + 1\right]$
- $T = \frac{1}{2}m[\dot{x}^2 + \dot{y}^2] = \frac{1}{2}mR^2\omega^4 t^2$

$\rightarrow$ particle accelerator