LECTURE 13: SIMULTANEOUS EQUATIONS: NO SOLUTIONS OR AN INFINITE NUMBER. CHANGE OF BASIS.

Prof. N. Harnew
University of Oxford

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Outline: 13. SIMULTANEOUS EQUATIONS: NO SOLUTIONS OR AN INFINITE NUMBER. CHANGE OF BASIS.

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13.1 Solutions do not exist

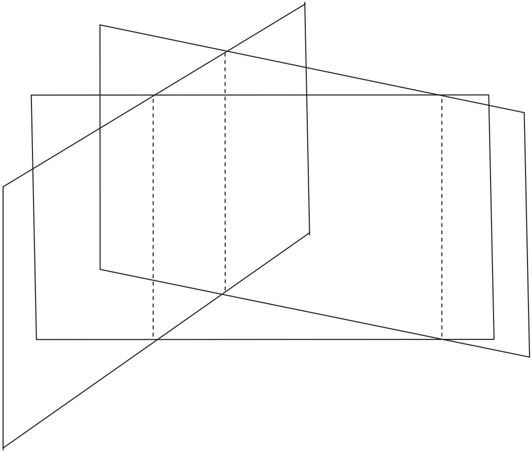
- Recall: 3 simultaneous linear equations:

$$\begin{array}{ccccccc} a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & = & b_2 \\ a_{31}x_1 & + & a_{32}x_2 & + & a_{33}x_3 & = & b_3 \end{array} \quad (1)$$

$$\text{i.e.} \quad A\mathrm{x} = b, \qquad \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \quad (2)$$

- Solutions do not exist if:
  - $|A| = 0$ and $\underline{\mathbf{b}} \neq 0$ and
    [Rank of coefficient matrix] $<$ [Rank of augmented matrix]
  - i.e. $|A| = 0$ and any of Cramer's determinants are not equal to zero (*)
    (*) since it is the Cramer's determinants (either $= 0$ or $\neq 0$) which then determine the rank of the augmented matrix.

13.1.1 Case 1



- Lines of intersection of the planes are parallel to each other.

Case 1 : Example

$$\begin{array}{ccccccc} 2x & + & 3y & + & 4z & = & 1 \\ x & + & 2y & + & 2z & = & 2 \\ -x & + & y & - & 2z & = & 3 \end{array} \tag{3}$$

$$|A| = \begin{vmatrix} 2 & 3 & 4 \\ 1 & 2 & 2 \\ -1 & 1 & -2 \end{vmatrix} = (2 \times -6) + 0 + (4 \times 3) = 0 \tag{4}$$

- Rank of coefficient matrix = 2

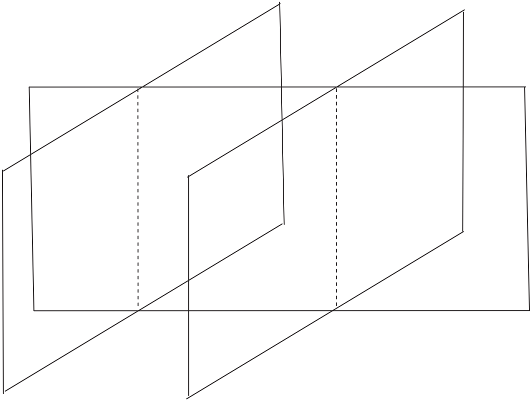
$$\text{Get rank of augmented matrix} \quad \left(\begin{array}{ccc|c} 2 & 3 & 4 & 1 \\ 1 & 2 & 2 & 2 \\ -1 & 1 & -2 & 3 \end{array} \right) \tag{5}$$

$$\text{First Cramer's determinant :} \quad \begin{vmatrix} 1 & 3 & 4 \\ 2 & 2 & 2 \\ 3 & 1 & -2 \end{vmatrix} = -6 + (3 \times 10) + (4 \times -4) = 8 \neq 0 \tag{6}$$

- Rank of augmented matrix = 3
  Hence, [Rank of coefficient matrix] $<$ [Rank of augmented matrix]

No solution

13.1.2 Case 2



- Two of the planes are parallel making the equations inconsistent.

Case 2 : Example

$$\begin{array}{ccccccccc} 2x & + & 3y & + & 4z & = & 1 & & (1) \\ x & + & 2y & + & 2z & = & 2 & & (2) \\ 4x & + & 6y & + & 8z & = & 10 & & (3) \end{array} \qquad (7)$$

- By inspection, planes defined by Equ's (1) and (3) are parallel with different perpendicular distances from origin.
- Inconsistent equations, no solution is possible.

$$|A| = \begin{vmatrix} 2 & 3 & 4 \\ 1 & 2 & 2 \\ 4 & 6 & 8 \end{vmatrix} = 2 \times 4 + 0 + 4 \times -2 = 0 \qquad (8)$$

$$\text{First Cramer's determinant} = \begin{vmatrix} 1 & 3 & 4 \\ 2 & 2 & 2 \\ 10 & 6 & 8 \end{vmatrix} \neq 0 \qquad (9)$$

13.2 An infinite number of solutions

13.2.1 Case 1

$$Ax = b$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \qquad (10)$$

An infinite number of solutions if:

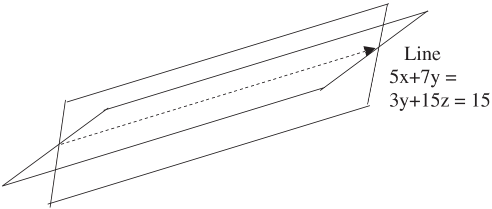
- $|A| = 0$ and $\underline{\mathbf{b}} \neq 0$ and
  [Rank of coefficient matrix] $=$ [Rank of augmented matrix]
- i.e. $|A| = 0$ and Cramer's determinants are all zero.

Then EITHER -

One of the equations is not unique:

Example
$$\begin{array}{ccccccccc} 2x & + & 3y & + & z & = & 7 & (1) \\ x & + & 2y & + & 3z & = & 6 & (2) \\ 4x & + & 6y & + & 2z & = & 14 & (3) \end{array} \qquad (11)$$

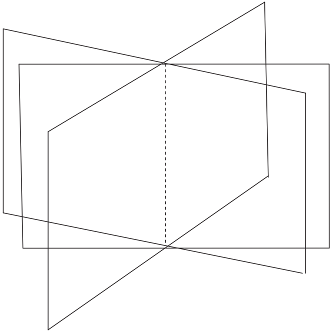
- 3 unknowns, Equ (3) is multiple of Equ (1) and is redundant $\Rightarrow$ system is under-constrained.



- These equations represent the intersection of two planes, with a common line $5x + 7y = 15 = 3y + 15z$. Hence infinite number of solutions.

OR

All three planes meet on a common line:



Example

$$\begin{array}{ccccccc} 2x & + & 3y & + & 4z & = & 2 \\ x & + & 2y & + & 2z & = & 1 \\ -x & + & y & - & 2z & = & -1 \end{array}$$

- $|A| = \begin{vmatrix} 2 & 3 & 4 \\ 1 & 2 & 2 \\ -1 & 1 & -2 \end{vmatrix}$
  $= 2 \times (-6) - 3 \times 0 + 4 \times 3 = 0$
- 3rd Cramer's determinant
  $\Delta_z = \begin{vmatrix} 2 & 3 & 2 \\ 1 & 2 & 1 \\ -1 & 1 & -1 \end{vmatrix}$
  $= 2 \times (-3) - 3 \times 0 + 2 \times 3 = 0$
- Also $\Delta_x = \Delta_y = 0$

- All three planes meet on a common line because:
  - $|A| = 0$
  - All Cramer's determinants are zero
  - AND all the equations are unique.

13.2.2 Case 2: Homogeneous equations

- $|A| = 0$ and $\underline{\mathbf{b}} = 0$

$$\begin{array}{ccccccc} a_{11}x & + & a_{12}y & + & a_{13}z & = & 0 \\ a_{21}x & + & a_{22}y & + & a_{23}z & = & 0 \\ a_{31}x & + & a_{32}y & + & a_{33}z & = & 0 \end{array}$$

- $\underline{\mathbf{b}} = 0$ gives the trivial solution $(x, y, z) = (0, 0, 0)$ *unless* $|A| = 0$
- Three planes meet on a common line passing through the origin. Only the *ratios* $x/y,\ x/z,\ y/z$ can be found.
- Example

$$\begin{array}{ccccccccc} 2x & + & 3y & + & 4z & = & 0 & & (1) \\ x & + & 2y & + & 2z & = & 0 & & (2) \\ -x & + & y & - & 2z & = & 0 & & (3) \end{array}$$

  - $|A| = 0$ and $\underline{\mathbf{b}} = 0$
  - Intersection: a line through the origin $\rightarrow$ $y = 0, x = -2z$
    [Trivially found $\rightarrow$ add Eq (2) & (3), then substitute for y.]

13.3 CHANGE OF BASIS

- A general vector $|\mathbf{x}\rangle$ in N-dimensional vector space is represented by a sum of base vectors $|\mathbf{e_i}\rangle$

  $$|\mathbf{x}\rangle = \sum_{i=1}^{N} x_i|\mathbf{e_i}\rangle \qquad \rightarrow \text{Equ (*)}$$

  where x_i are the coefficients of $|\mathbf{x}\rangle$ w.r.t the basis.

- Consider a change of basis vectors to $|\mathbf{e'_j}\rangle$ (e.g. transformation to a different coordinate system)

- The prime basis will be related to the un-primed basis by a set of transformation coefficients S_{ij}

  $$|\mathbf{e'_j}\rangle = \sum_{i=1}^{N} S_{ij}|\mathbf{e_i}\rangle$$

- The vector must be the same (i.e. length, direction) whether its represented in basis $|\mathbf{e_i}\rangle$ or $|\mathbf{e_i'}\rangle$, hence

$$\sum_{i=1}^{N} x_i |\mathbf{e_i}\rangle = \sum_{j=1}^{N} x_j' |\mathbf{e_j'}\rangle = \sum_{j=1}^{N} x_j' \sum_{i=1}^{N} S_{ij} |\mathbf{e_i}\rangle$$

  where x_j' are the components of $|\mathbf{x}\rangle$ in basis $|\mathbf{e}'\rangle$.
- Since the x_j' and S_{ij} are just numbers, we can change order of summation:

$$|\mathbf{x}\rangle = \sum_{i=1}^{N} \sum_{j=1}^{N} S_{ij} x_j' \, |\mathbf{e_i}\rangle$$

- Comparing with Equ (*) : $\left(i.e. \ |\mathbf{x}\rangle = \sum_{i=1}^{N} x_i |\mathbf{e_i}\rangle \right)$

$$\rightarrow \quad \boxed{x_i = \sum_{j=1}^{N} S_{ij} x_j'}$$

  Hence the components of the vector transform in the same way, but *inversely*, to the base vectors themselves (which ensures $|\mathbf{x}\rangle$ is unchanged by the transformation).

13.3.1 Change of basis in matrix form

- We can write the transformation of the base vectors in matrix form:

$$|\mathbf{e}'\rangle = S|\mathbf{e}\rangle$$

  where S is the *transformation matrix*.

- Since the basis vectors are linearly independent $det(S)$ and hence S^{-1} will exist.
  Hence $S^{-1}|\mathbf{e}'\rangle = S^{-1}S|\mathbf{e}\rangle \rightarrow$ $|\mathbf{e}\rangle = S^{-1}|\mathbf{e}'\rangle$

- This is the inverse transformation which transforms $|\mathbf{e}'\rangle$ back to $|\mathbf{e}\rangle$, leaving the base vectors unchanged by the two successive transformations.

- Conversely vector x in $|\mathbf{e}\rangle$ is represented in $|\mathbf{e}'\rangle$ by

  $x' = S^{-1}x$ with inverse transform $x = Sx'$