Classical Mechanics

LECTURE 12:

MORE ON ROCKETS; NON-INERTIAL FRAMES

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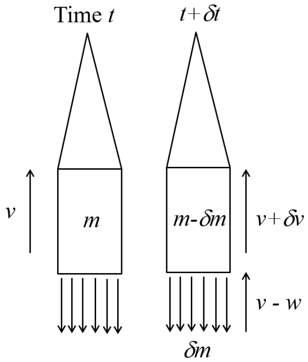
OUTLINE : 12. MORE ON ROCKETS; NON-INERTIAL FRAMES

12.1 The rocket : vertical launch

- Rocket equation:
  $m\frac{dv}{dt} + w\frac{dm}{dt} = F$
- Rocket rises against gravity
  $F = -mg$
- Mass is ejected at *constant* velocity w relative to the rocket
- Rocket ejects mass uniformly:
  $m = m_0 - \alpha t$
  $\rightarrow \frac{dm}{dt} = -\alpha$



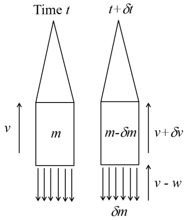
- Now consider upward motion:
- $mdv = (-mg + w\alpha)dt \quad \rightarrow \quad \int_{v_i}^{v_f} dv = \int_{t_i}^{t_f} \left(-g + \frac{w\alpha}{m_0 - \alpha t}\right) dt$
- $v_f - v_i = \left[-g(t_f - t_i) - w\log_e \frac{(m_0 - \alpha t_f)}{(m_0 - \alpha t_i)}\right]$
  $= [-g(t_f - t_i) - w\log_e(m_f/m_i)]$

Rocket vertical launch, continued

The rocket starts from rest at $t = 0$; half the mass is fuel. What is the velocity and height reached by the rocket at burn-out at time $t = T$?

- $v = \left[-gt - w \log_e \frac{(m_0 - \alpha t)}{(m_0)} \right] = \left[-gt - w \log_e \left(1 - \frac{\alpha}{m_0} t \right) \right] = \frac{dx}{dt}$
- What is the condition for the rocket to rise ? $\rightarrow \frac{dv}{dt} > 0$

  At $t = 0$, $m = m_0$, $\frac{dm}{dt} = -\alpha$: $\alpha w - m_0 g > 0 \rightarrow w > \frac{m_0 g}{\alpha}$



- $m = m_0 - \alpha t$; at burnout $t = T$, $m = \frac{m_0}{2} \rightarrow \alpha = \frac{m_0}{2T}$
- Maximum velocity is at the burn-out of the fuel:

  At $t = T$: $v_{max} = -gT + w \log_e 2$

  Height : $\int_0^x dx = \int_0^T \left[-gt - w \log_e \left(1 - \frac{\alpha}{m_0} t \right) \right] dt$

  - Standard integral : $\int \log_e z \, dz = z \log_e z - z$
  - $x = -\frac{gT^2}{2} + \frac{w\, m_o}{\alpha} \left[\left(1 - \frac{\alpha}{m_0} t \right) \left(\log_e \left(1 - \frac{\alpha}{m_0} t \right) \right) - \left(1 - \frac{\alpha}{m_0} t \right) \right]_0^T$
  - After simplification : $x = -\frac{gT^2}{2} + wT(1 - \log_e 2)$

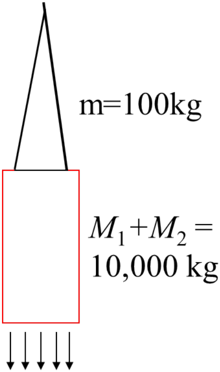
12.2 The 1-stage vs. 2-stage rocket

A two stage rocket is launched vertically from earth, total mass $M_0 = 10000$ kg and carries an additional payload of $m = 100$ kg. The fuel is 75% of the mass in both stages; burn rate $\alpha = 500$ kg s^{-1}, thrust velocity $w = 2.5$ km s^{-1}. The mass of the 2nd stage is 900 kg.

i) Calculate the final speed for the equivalent single stage rocket

ii) Find the final speed of the 2-stage rocket

(i) Single stage rocket :



- Time to burn-out :

$$\left|\frac{\Delta m}{\Delta t}\right| = \alpha \rightarrow T = 0.75(M_1 + M_2)/500 \text{ [kg s}^{-1}\text{]}$$

- From before : $v_f = -gT + w \log_e(m_i/m_f)$

$$m_i = M_1 + M_2 + m \;\; ; \;\; m_f = 0.25(M_1 + M_2) + m$$

- Single stage : $v_f = 3.25\,\text{km s}^{-1}$
- Earth's escape speed :

$$\frac{1}{2}mv^2 > \frac{GM_e m}{R_E} \quad \rightarrow \quad v_{esc} = \sqrt{\frac{2GM_E}{R_E}}$$

$$\rightarrow \quad v_{esc} = 11.2\,\text{km s}^{-1} \;\; (i.e.\ v_f < v_{esc})$$

(ii) The 2-stage rocket

Stage 1

- $|\frac{\Delta m}{\Delta t}| = \alpha \quad \rightarrow T = 0.75 M_1/500$ [kg s^{-1}]
- $v_1 = -gT + w \log_e(m_i/m_f)$

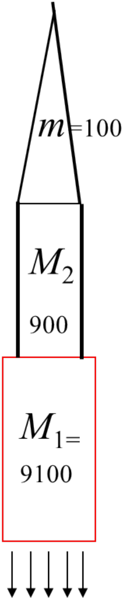
  $m_i = M_1 + M_2 + m \; ; \;\; m_f = 0.25 M_1 + M_2 + m$
- After first stage :$v_f = 2.68$ km s^{-1}

Stage 2

- $|\frac{\Delta m}{\Delta t}| = \alpha \quad \rightarrow T = 0.75 M_2/500$ [kg s^{-1}]
- $v_2 = v_1 - gT + w \log_e(m_i/m_f)$

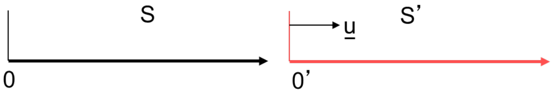
  $m_i = M_2 + m \; ; \;\; m_f = 0.25 M_2 + m$
- After second stage :

  $v_2 = 2.68 + 2.80 = 5.48$ km s^{-1}
- Need a 3rd stage, more thrust or less payload to escape



12.3 Non-inertial reference frames

A frame in which Newton's first law is not satisfied - the frame is accelerating (i.e. subject to an external force)

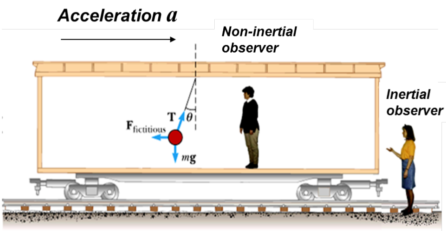


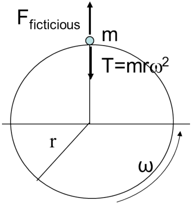
Recall Galilean transformation but now $\underline{\mathbf{u}}$ varies with time :

	Galilean transformation	Accelerating frame
Position	$\underline{\mathbf{r}}' = \underline{\mathbf{r}} - \underline{\mathbf{u}}t$	$\underline{\mathbf{r}}' = \underline{\mathbf{r}} - \int \underline{\mathbf{u}}(t)\,dt$
Velocity	$\underline{\mathbf{v}}' = \underline{\mathbf{v}} - \underline{\mathbf{u}}$	$\underline{\mathbf{v}}' = \underline{\mathbf{v}} - \underline{\mathbf{u}}(t)$
Acceleration	$\frac{d\underline{\mathbf{v}}'}{dt} = \frac{d\underline{\mathbf{v}}}{dt}$	$\frac{d\underline{\mathbf{v}}'}{dt} = \frac{d\underline{\mathbf{v}}}{dt} - \frac{d\underline{\mathbf{u}}}{dt}$
Force on mass	$\underline{\mathbf{F}}'(\underline{\mathbf{r}}') = F(\underline{\mathbf{r}}) = m\frac{d\underline{\mathbf{v}}}{dt}$	$\underline{\mathbf{F}}'(\underline{\mathbf{r}}') = \underline{\mathbf{F}}(\underline{\mathbf{r}}) - m\frac{d\underline{\mathbf{u}}}{dt}$

- So even if $F(\underline{\mathbf{r}}) = 0$, from NII, there is an *apparent (or "ficticious") force* acting in S' of $\underline{\mathbf{F}}'(\underline{\mathbf{r}}') = -m\frac{d\underline{\mathbf{u}}}{dt}$

12.3.1 Commonplace examples





Accelerating train

- In the inertial frame

  $\sum F_x = T \sin\theta = ma$

  $\sum F_y = T \cos\theta - mg = 0$

- In the non-inertial frame

  $\sum F'_x = T \sin\theta - F_{fictitious}$

  $\sum F'_y = T \cos\theta - mg = 0$

In NIF need to introduce $F_{fictitious} = ma$ to explain the displacement of the bob.

Mass rotating in a circle

- In the inertial frame

  Centripetal acceleration provided by tension in the string

  $T = mr\omega^2$

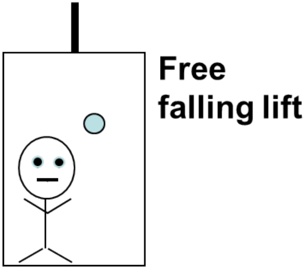
- In the non-inertial frame

  The block is at rest and its acceleration is zero

In NIF need $F_{fictitious} = mr\omega^2$ (centrifugal force) to balance the tension in the string.

12.3.2 Example : accelerating lift

- First consider the lift in free-fall
- The ball is "weightless" (stationary or moves at constant velocity) according to an observer in the lift $\rightarrow$ lift becomes an inertial frame (like in deep space) : NI.
- To this observer the fictitious acceleration $\left(-\frac{d\mathbf{u}}{dt}\right)$ balances the gravitational acceleration



Observer in accelerating lift

- Lift plus passenger (total mass M) is now accelerated upwards with force F. Passenger drops ball mass m' from height h $(M >> m')$.
- Total force on lift $F_{tot} = F - Mg = Ma$

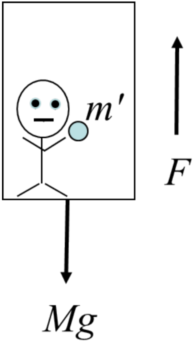
  Acceleration of lift $a = \frac{F}{M} - g$
- According to passenger in lift frame, *downwards* acceleration of ball is

  $$= (\frac{F}{M} - g) + g = \frac{F}{M} \text{ (downwards)}$$
- Hence weight of ball appears to passenger to be $(F/M) \times m'$
- Check this:

  If $F = 0$, free-fall, ball is weightless

  If $F = Mg$, lift is stationary, ball has weight $m'g$

  - Time for ball to reach floor, use $h = \frac{1}{2}at^2 \rightarrow t = \sqrt{\frac{2hM}{F}}$



Observer watching from a frame outside the lift

- Observer watches ball fall with acceleration g and the lift rise with acceleration $\frac{F}{M} - g$

- Equate times when ball reaches floor:

$$t = \underbrace{\sqrt{\frac{2h'}{g}}}_{\text{ball falling}} = \underbrace{\sqrt{\frac{2(h-h')}{F/M-g}}}_{\text{lift rising}}$$

- Solve for h' $\rightarrow$ $h' = \frac{Mgh}{F}$

- Substitute back
$\rightarrow \; t = \sqrt{\frac{2Mh}{F}}$ as before.