LECTURE 12:

SOLUTIONS TO SIMULTANEOUS LINEAR EQUATIONS

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12.1 Methods used to solve for unique solution

We will consider 3 methods to solve the system of equations with a unique solution:

- Matrix inverse method
- Cramer's method
- Gauss reduction method

12.1.1 Matrix inversion method

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \cdots \\ b_n \end{pmatrix} \quad (1)$$

- The equations are written $A\mathrm{x} = b$, therefore we write $\mathrm{x} = A^{-1}b$ where $(A^{-1})_{ij} = (C^T)_{ij}/|A|$ as before.
- This has the advantage that the solutions x_i can be trivially found, once A^{-1} has been calculated. But A^{-1} can be cumbersome to calculate.
- Note the following:
  - The method needs $|A|$ to be $\neq 0$ (i.e. non-singular),
  - If all the $b_i = 0$, only the trivial solution $x_i = 0$ will be found.

Example: 3×3 *matrix*

$$\begin{array}{rcrcrcr} x & + & y & + & z & = & 6 \\ 2x & - & y & + & z & = & 3 \\ -3x & - & 2y & + & 2z & = & -1 \end{array} \tag{2}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ -3 & -2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \\ -1 \end{pmatrix} \tag{3}$$

- Calculate determinant of A:

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ -3 & -2 & 2 \end{vmatrix} = (1 \times 0) - (1 \times 7) + (1 \times -7) = -14 \tag{4}$$

  $|A| \neq 0$ therefore there exists a unique solution
- Form the matrix of cofactors

$$C = \begin{pmatrix} 0 & -7 & -7 \\ -4 & 5 & -1 \\ 2 & 1 & -3 \end{pmatrix} \tag{5}$$

- Now take the transpose:

$$C^T = \begin{pmatrix} 0 & -4 & 2 \\ -7 & 5 & 1 \\ -7 & -1 & -3 \end{pmatrix} \tag{6}$$

- The inverse is then $C^T/|A|$

$$A^{-1} = \frac{-1}{14} \times \begin{pmatrix} 0 & -4 & 2 \\ -7 & 5 & 1 \\ -7 & -1 & -3 \end{pmatrix} \tag{7}$$

- Then evaluate the matrix $\quad \mathrm{x} = A^{-1}b$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{-1}{14} \times \begin{pmatrix} 0 & -4 & 2 \\ -7 & 5 & 1 \\ -7 & -1 & -3 \end{pmatrix} \begin{pmatrix} 6 \\ 3 \\ -1 \end{pmatrix} \tag{8}$$

$$x = (0 \times 6 + -4 \times 3 + 2 \times -1)/(-14) = 1$$
$$y = (-7 \times 6 + 5 \times 3 + 1 \times -1)/(-14) = 2$$
$$z = (-7 \times 6 + -1 \times 3 + -3 \times -1)/(-14) = 3 \quad \text{Q.E.D.}$$

12.1.2 Cramer's method

- Cramer's method is illustrated for a 3×3 system by writing the linear equations in *vector* form:

$$\begin{array}{ccccccc} a_1 x & + & b_1 y & + & c_1 z & = & v_1 \\ a_2 x & + & b_2 y & + & c_2 z & = & v_2 \\ a_3 x & + & b_3 y & + & c_3 z & = & v_3 \end{array} \quad \Rightarrow \quad x\underline{\mathbf{a}} + y\underline{\mathbf{b}} + z\underline{\mathbf{c}} = \underline{\mathbf{v}} \qquad (9)$$

where $\underline{\mathbf{a}} = (a_1, a_2, a_3)$, $\underline{\mathbf{b}} = (b_1, b_2, b_3)$,
$\underline{\mathbf{c}} = (c_1, c_2, c_3)$, $\underline{\mathbf{v}} = (v_1, v_2, v_3)$.

- Define a vector perpendicular to $\underline{\mathbf{b}}$ and $\underline{\mathbf{c}}$ $\Rightarrow$ $\underline{\mathbf{n_x}} = \underline{\mathbf{b}} \times \underline{\mathbf{c}}$
- Take the dot product with the vector equation:

$$\underline{\mathbf{n_x}}.(x\underline{\mathbf{a}} + y\underline{\mathbf{b}} + z\underline{\mathbf{c}}) = \underline{\mathbf{n_x}}.\underline{\mathbf{v}}$$

- Remember that $(\underline{\mathbf{b}} \times \underline{\mathbf{c}}).\underline{\mathbf{b}} = (\underline{\mathbf{b}} \times \underline{\mathbf{c}}).\underline{\mathbf{c}} = 0$,

hence $x\underline{\mathbf{n_x}}.\underline{\mathbf{a}} = \underline{\mathbf{n_x}}.\underline{\mathbf{v}} \quad \Rightarrow \quad x = \dfrac{\underline{\mathbf{n_x}}.\underline{\mathbf{v}}}{\underline{\mathbf{n_x}}.\underline{\mathbf{a}}}$ Equ (*)

- Analogously, to find y : $\quad \underline{\mathbf{n_y}} = \underline{\mathbf{c}} \times \underline{\mathbf{a}}; \quad y = \dfrac{\underline{\mathbf{n_y}}.\underline{\mathbf{v}}}{\underline{\mathbf{n_y}}.\underline{\mathbf{b}}}$.

  to find z : $\quad \underline{\mathbf{n_z}} = \underline{\mathbf{a}} \times \underline{\mathbf{b}}; \quad z = \dfrac{\underline{\mathbf{n_z}}.\underline{\mathbf{v}}}{\underline{\mathbf{n_z}}.\underline{\mathbf{c}}}$.

- Come back to Eqn (*), the denominator is the triple scalar product:

$$\underline{\mathbf{n_x}}.\underline{\mathbf{a}} = \underline{\mathbf{a}}.(\underline{\mathbf{b}} \times \underline{\mathbf{c}}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \tag{10}$$

- But $det(A) = det(A^T)$ therefore

$$det(A) = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad \text{which is the } det \text{ of the coefficient matrix.} \tag{11}$$

- Also numeratator

$$\underline{\mathbf{n_x}}.\underline{\mathbf{v}} = \underline{\mathbf{v}}.(\underline{\mathbf{b}} \times \underline{\mathbf{c}}) = \begin{vmatrix} v_1 & v_2 & v_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} v_1 & b_1 & c_1 \\ v_2 & b_2 & c_2 \\ v_3 & b_3 & c_3 \end{vmatrix} \tag{12}$$

- Hence $x = \Delta_x/|A|$ (where $\Delta_x \equiv$ Cramer's determinant)
- Hence Cramer's determinants are simply $|A|$ with columns 1→3 replaced by the RHS of original set of equations, $\underline{\mathbf{v}}$, i.e.

$$x = \Delta_x/|A|, \quad y = \Delta_y/|A|, \quad z = \Delta_z/|A|$$

Example: same 3×3 *matrix*

$$\begin{array}{ccccccc} x & + & y & + & z & = & 6 \\ 2x & - & y & + & z & = & 3 \\ -3x & - & 2y & + & 2z & = & -1 \end{array} \Rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ -3 & -2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \\ -1 \end{pmatrix}$$

- Determinant of A:

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ -3 & -2 & 2 \end{vmatrix} = (1 \times 0) - (1 \times 7) + (1 \times -7) = -14 \quad (13)$$

$$\Delta_x = \begin{vmatrix} 6 & 1 & 1 \\ 3 & -1 & 1 \\ -1 & -2 & 2 \end{vmatrix} = (6 \times 0) - (1 \times 7) + (1 \times -7) = -14 \quad (14)$$

- Then $x = \Delta_x / |A| = (-14)/(-14) = 1$

$$\Delta_y = \begin{vmatrix} 1 & 6 & 1 \\ 2 & 3 & 1 \\ -3 & -1 & 2 \end{vmatrix} = (1 \times 7) - (6 \times 7) + (1 \times 7) = -28 \quad (15)$$

- and $y = \Delta_y / |A| = (-28)/(-14) = 2$

$$\Delta_z = \begin{vmatrix} 1 & 1 & 6 \\ 2 & -1 & 3 \\ -3 & -2 & -1 \end{vmatrix} = (1 \times 7) - (1 \times 7) + (6 \times -7) = -42 \quad (16)$$

- and $z = \Delta_z / |A| = (-42)/(-14) = 3$ Q.E.D.

12.1.3 Gauss reduction method: echelon form of matrix

Also called lower and upper triangle (echelon) decomposition.

- An $m \times n$ matrix is in *echelon* (or stair step or upper triangle) form if all the elements below the leading diagonal are zero:

$$\text{i.e.}\quad a_{ij} = 0 \;\; \text{for } i > j$$

- Example for an $m \times n$ matrix:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1m} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & \cdots & a_{2m} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3m} & \cdots & a_{3n} \\ 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a_{mm} & \cdots & a_{mn} \end{pmatrix} \tag{17}$$

- The matrix elements on the stair steps are called the *pivots* (in this case $a_{11} \to a_{mm}$).
- An $m \times n$ matrix is said to be in in *reduced echelon* (or reduced stair step) form if all the elements below *and above* the leading diagonal are zero, and the pivots are all ones.

Example of Gauss reduction method

The Gauss method reduces the augmented matrix to echelon form. The equations are manipulated and the matrix is used for "book-keeping" the coefficients.

$$\begin{array}{rcrcrcr} x & + & y & + & z & = & 6 \\ 2x & - & y & + & z & = & 3 \\ -3x & - & 2y & + & 2z & = & -1 \end{array} \tag{18}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ -3 & -2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \\ -1 \end{pmatrix} \tag{19}$$

- Start with the augmented matrix. Aim is to get the matrix in echelon form.

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 2 & -1 & 1 & 3 \\ -3 & -2 & 2 & -1 \end{array}\right) \tag{20}$$

- Multiply Row 1 by 2. Subtract Row 2 and put the answer in Row 2.

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 3 & 1 & 9 \\ -3 & -2 & 2 & -1 \end{array}\right) \tag{21}$$

- Multiply Row 1 by 3. Add Row 3 and put the answer in Row 3.

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 3 & 1 & 9 \\ 0 & 1 & 5 & 17 \end{array}\right) \tag{22}$$

- Multiply Row 3 by 3. Subtract from Row 2 and put the answer in Row 3.

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 3 & 1 & 9 \\ 0 & 0 & -14 & -42 \end{array}\right) \Rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & -14 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 9 \\ -42 \end{pmatrix} \tag{23}$$

- This is now in echelon form.

Substitute back

- From Row 3: $-14z = -42 \quad \rightarrow z = 3$
- From Row 2: $3y + z = 9 \quad \rightarrow y = 2$
- From Row 1: $x + y + z = 6 \quad \rightarrow x = 1$

It's tedious, but can also obtain the reduced echelon form

- Divide Row 3 by -14.

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 3 & 1 & 9 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

- Subtract Row 3 from Row 1, put the answer in Row 1.

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & 3 & 1 & 9 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

- Subtract Row 3 from Row 2, put the answer in Row 2.

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & 3 & 0 & 6 \\ 0 & 0 & 1 & 3 \end{array}\right) \qquad (24)$$

- Multiply Row 1 by 3, subtract Row 2, put the answer in Row 1.

$$\left(\begin{array}{ccc|c} 3 & 0 & 0 & 3 \\ 0 & 3 & 0 & 6 \\ 0 & 0 & 1 & 3 \end{array}\right) \qquad (25)$$

- Divide Row 1 by 3, Divide Row 2 by 3.

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right) \quad \text{Gives the same result!} \qquad (26)$$

12.2 Gauss reduction for finding the matrix inverse

- An aside - Example : Construct the extended augmented matrix. Perform *exactly* the same operations as before to get reduced echelon on the RHS:

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & -1 & 1 & 0 & 1 & 0 \\ -3 & -2 & 2 & 0 & 0 & 1 \end{array}\right) \tag{27}$$

- Multiply Row 1 by 2. Subtract Row 2 and put the answer in Row 2.

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 3 & 1 & 2 & -1 & 0 \\ -3 & -2 & 2 & 0 & 0 & 1 \end{array}\right) \tag{28}$$

.... etc etc. Reduce to echelon form. Finishing with ....

- Divide Row 1 by 3 & row 2 by 3.

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 4/14 & -2/14 \\ 0 & 1 & 0 & 1/2 & -5/14 & -1/14 \\ 0 & 0 & 1 & 7/14 & 1/14 & 3/14 \end{array}\right) \tag{29}$$

- Gives the inverse of the matrix to be:

$$A^{-1} = \frac{1}{14}\begin{pmatrix} 0 & 4 & -2 \\ 7 & -5 & -1 \\ 7 & 1 & 3 \end{pmatrix} \tag{30}$$