Classical Mechanics

LECTURE 11:

NEWTON'S SECOND LAW AND VARIABLE MASS

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OUTLINE : 11. NEWTON'S SECOND LAW AND VARIABLE MASS

Intro : programme for Hilary term (20 lectures)

11.1 Variable mass : a body acquiring mass

11.2 Example - the raindrop

11.3 Ejecting mass : the rocket equation

11.4 The rocket : horizontal launch

Programme for Hilary term (20 lectures)

- Lectures 1-5

  Rocket motion. Motion in B and E fields
- Lectures 6-10

  Central forces (orbits)
- Lectures 11-15

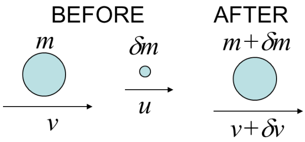
  Rotational dynamics (rigid body etc)
- Lectures 16-20

  Lagrangian dynamics

- Plus 4 problem sets for your enjoyment

11.1 Variable mass : a body acquiring mass

- A body of mass m has velocity v. In time δt it it acquires mass δm, which is moving along v direction with velocity u

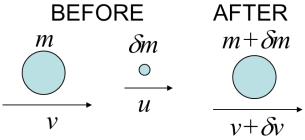


- The change in mass m is $m + \delta m$, the change in velocity v is $v + \delta v$
  - Case 1: No external force. Change of momentum Δp

$$\Delta p = \underbrace{(m + \delta m)(v + \delta v)}_{\textit{After}} - \underbrace{(mv + u\delta m)}_{\textit{Before}} = 0$$

  - $mv + m\delta v + v\delta m + \underbrace{\delta m \delta v}_{\textit{Ignore}} - mv - u\delta m = m\delta v + (v - u)\,\delta m = 0$
  - Divide by δt (time over which mass acquisition occurs) :
    $\frac{\Delta p}{\delta t} = m\frac{\delta v}{\delta t} + \underbrace{(v - u)}_{\textit{Relative velocity}, \, w} \frac{\delta m}{\delta t} = 0$
  - As $\delta t \to 0$, $m\frac{dv}{dt} + w\frac{dm}{dt} = 0$ (in this case $\frac{dv}{dt}$ is -ve as expected)

A body acquiring mass - with external force



- Case 2: Application of an external force F
- NII : change of momentum $= \Delta p = F\delta t = m\delta v + w\,\delta m$ as before, where $w = (v - u)$
- Divide by δt and let $\delta t \to 0$

$$\boxed{m\frac{dv}{dt} + w\frac{dm}{dt} = F}$$

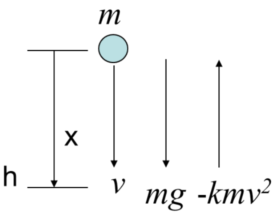
Note : ONLY in the case when $u = 0$ does $\frac{d}{dt}(mv) = F$

11.2 Example - the raindrop

An idealised raindrop has initial mass m_0, is at height h above ground and has zero initial velocity. As it falls it acquires water (added from rest) such that its increase in mass at speed v is given by $dm/dt = bmv$ where b is a constant. The air resistance is of the form kmv^2 where k is a constant.

- Formulate the equation of motion :
- $m\frac{dv}{dt} + w\frac{dm}{dt} = F$

  $\rightarrow \quad m\frac{dv}{dt} + w\frac{dm}{dt} = mg - kmv^2$
- $w = v$ (since $u = 0$) ; $\frac{dm}{dt} = bmv$
- $\frac{dv}{dt} + (b+k)v^2 = g$



  - Terminal velocity :
  - $\frac{dv}{dt} = 0 \quad \rightarrow \quad v_T = \sqrt{\frac{g}{b+k}}$

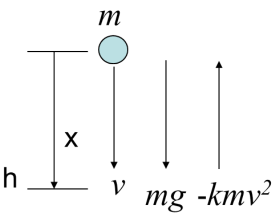
The raindrop, continued

- Calculate raindrop mass vs. distance
- $\frac{dm}{dt} = bmv$
- $\frac{dm}{dx} = \frac{dm}{dt}\frac{dt}{dx} = \frac{bmv}{v} = bm$

  $\rightarrow \quad \frac{dm}{m} = bdx$

  Integrate : $\quad [\log_e m]_{m_0}^{m} = [bx]_0^x$
- $m = m_0 \exp(bx)$

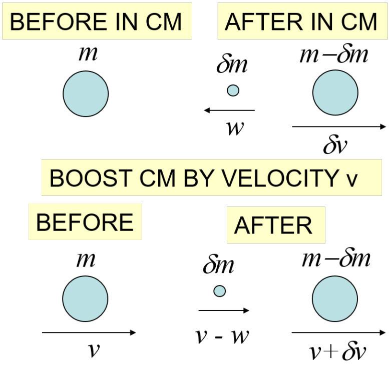
  (Mass grows exponentially with x)



  - What is its speed at ground level ?
  - $\frac{dv}{dt} + (b+k)v^2 = g \quad \rightarrow \quad \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$
  - $\int_0^{v_h} \frac{vdv}{g-(b+k)v^2} = \int_0^h dx \quad \rightarrow \quad h = \left[-\left(\frac{\log_e(g-(b+k)v^2)}{2(b+k)}\right)\right]_0^{v_h}$
  - Solving : $\boxed{v_h = \sqrt{\frac{g}{b+k}\left[1 - \exp\left(-2h(b+k)\right)\right]}}$

11.3 Ejecting mass : the rocket equation

- A body of mass m has velocity v. In time δt it *ejects* mass δm, with relative velocity w to the body



- Change of momentum $\Delta p = \underbrace{\delta m(v - w) + (m - \delta m)(v + \delta v)}_{After}$
  $- \underbrace{mv}_{Before}$
  $= v\delta m - w\delta m + mv + m\delta v -$
  $v\delta m - \underbrace{\delta m\delta v}_{Ignore} - mv$

- With external force : $F = \frac{\Delta p}{\delta t} = m\frac{\delta v}{\delta t} - w\frac{\delta m}{\delta t}$
- As $\delta t \to 0$, $\frac{\delta v}{\delta t} \to \frac{dv}{dt}$ & $\frac{\delta m}{\delta t} \to -\frac{dm}{dt}$ (as $\frac{\delta m}{\delta t}$ is +ve but $\frac{dm}{dt}$ is -ve)
- Hence, again, $\boxed{F = m\frac{dv}{dt} + w\frac{dm}{dt}}$ the rocket equation

11.4 The rocket : horizontal launch

- Rocket equation:
  $m\frac{dv}{dt} + w\frac{dm}{dt} = F = 0$ (no gravity)
- Assume mass is ejected with *constant* relative velocity to the rocket w
- $m\,dv = -w\,dm \quad \rightarrow \quad dv = -w\frac{dm}{m}$
- Initial/final velocity = v_i, v_f
  Initial/final mass = m_i, m_f
- $\int_{v_i}^{v_f} dv = -w\int_{m_i}^{m_f} \frac{dm}{m}$
- $v_f - v_i = w\log_e(m_i/m_f)$

  This expression gives the dependence of rocket velocity as a function of its mass