LECTURE 11: THE INVERSE AND RANK OF A MATRIX AND SIMULTANEOUS EQUATIONS IN MATRIX FORM

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Outline: 11. THE INVERSE AND RANK OF A MATRIX AND SIMULTANEOUS EQUATIONS IN MATRIX FORM

11.1 Prescription for finding the inverse of a matrix

For a square matrix A: $AA^{-1} = A^{-1}A = I$

Prescription to find A^{-1}:

1. Start from a square matrix A

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \quad (1)$$

2. Form the matrix of cofactors of A:

$$C = \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{pmatrix} \quad (2)$$

   where cofactor $C_{ij} = [\mathrm{minor}] \times [\mathrm{sign}] = \mathrm{M_{ij}} \times (-1)^{(\mathrm{i+j})}$ as before.

3. Take the transpose $C \Rightarrow C^T$ (the adjugate matrix)
4. Divide by the determinant of A.

Then the elements of A^{-1} are $(A^{-1})_{ik} = (C^T)_{ik}/|A| = C_{ki}/|A|$

If $|A| = 0$, the matrix the matrix has no inverse (i.e. singular).

11.2 Proof for a general 3×3 matrix

- Start from $A^{-1}A = I$
- Taking a general element of the LHS $(A^{-1}A)_{ij} = \sum_k (A^{-1})_{ik} A_{kj}$
- Now we *assume* the postulate $(A^{-1})_{ik} = (C^T)_{ik}/|A|$
  - Hence $(A^{-1}A)_{ij} = \boxed{\sum_k \frac{C_{ki}}{|A|} A_{kj} = I_{ij} = \delta_{ij}}$ $\rightarrow$ Eqn (*)

    where δ_{ij} is the Kronecker delta
    ($\delta_{ij} = 1$ for $i = j$; $= 0$ for $i \neq j$)
- What we are going to do from here: Consider a general 3×3 matrix A and show that Equ (*) is satisfied for both diagonal and off-diagonal elements in turn.

The diagonal elements

- Take the general 3×3 matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \quad (3)$$

- Referring to Equ (*), consider $\sum_k C_{ki} A_{kj} / |A|$ for diagonal elements $i = j$. Take as a specific example $i = j = 1$

$$\frac{\sum_k C_{k1} A_{k1}}{|A|} = \left(C_{11} a_{11} + C_{21} a_{21} + C_{31} a_{31} \right) / |A|$$

- But $\sum_k C_{k1} A_{k1}$ is the *determinant* from the first column

$$= a_{11} \times \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21} \times \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31} \times \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} = |A| \quad (4)$$

- i.e. when $i = j = 1$, $\frac{\sum_k C_{ki} A_{ki}}{|A|} = \frac{|A|}{|A|} = 1 = \delta_{ii}$ (cf. Equ (*))
  Also true had we taken $i = j = 2$ or $i = j = 3$.

So the diagonal elements are consistent with the postulate.

The off-diagonal elements

- Next consider $\sum_k C_{ki}A_{kj}/|A|$ for $i \neq j$ in Equ (*):
- Take a specific example $(i = 1, j = 2)$
  $\sum_k C_{k1}A_{k2} = C_{11}a_{12} + C_{21}a_{22} + C_{31}a_{32}$

$$= a_{12} \times \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{22} \times \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{32} \times \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \quad (5)$$

- Next re-create the full determinant:

$$\begin{vmatrix} a_{12} & a_{12} & a_{13} \\ a_{22} & a_{22} & a_{23} \\ a_{32} & a_{32} & a_{33} \end{vmatrix} = 0 \ \text{[since columns 1 and 2 have equal elements]}. \quad (6)$$

- So $\sum_k C_{ki}A_{kj}/|A| = 0$ for $i \neq j$ (cf. Equ (*) $\Rightarrow \delta_{ij} = 0$).
  Also true had we taken any values $i \neq j$.

So the non-diagonal elements are also consistent with the postulate.

- We conclude that as a general result, $A^{-1} = C^T/|A|$

11.3 Inverse of 2 × 2 matrix

- Start with matrix A:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \tag{7}$$

  - The determinant is $|A| = (a_{11}a_{22} - a_{12}a_{21})$
  - The matrix of cofactors is:

$$C = \begin{pmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11} \end{pmatrix} \tag{8}$$

- The inverse of A is

$$A^{-1} = C^T/|A| = \frac{1}{(a_{11}a_{22} - a_{12}a_{21})} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix} \tag{9}$$

- This is general for a 2 × 2 matrix and is easy to remember.

11.3.1 Example

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \tag{10}$$

$$A^{-1} = \frac{1}{(1 \times 4 - 2 \times 3)} \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} \tag{11}$$

$$= -\frac{1}{2} \times \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix} \tag{12}$$

Check:

$$A^{-1}A = \begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \tag{13}$$

$$= \begin{pmatrix} -2 \times 1 + 1 \times 3 & -2 \times 2 + 1 \times 4 \\ 3/2 \times 1 - 1/2 \times 3 & 3/2 \times 2 - 4 \times 1/2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ as required.} \tag{14}$$

11.4 Rank of a matrix

- Two equivalent definitions:
  - The *rank* of an $m \times n$ matrix is defined as the number of *linear independent* rows or columns in the matrix (whichever is the smallest).
  - Alternatively the rank of an $m \times n$ matrix is equal to the size of the largest *square* sub-matrix that is contained in the m rows and n columns of the matrix whose determinant is non-zero.
- Hence the rank of matrix A is always $\leq$ to the *smaller* of m or n.
- For an $n \times n$ square matrix, $|A| = 0$ unless the rank = n.

11.4.1 Example

$$3 \times 3 \text{ matrix} \qquad A = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 2 \\ 4 & 1 & 3 \end{pmatrix} \tag{15}$$

$$|A| = \begin{vmatrix} 1 & 1 & 0 \\ 2 & 0 & 2 \\ 4 & 1 & 3 \end{vmatrix} = 1 \times (-2) - 1 \times (6 - 8) + 0 = 0 \tag{16}$$

However a number of sub-matrices are non-zero:

$$\begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} = -2, \quad \begin{vmatrix} 1 & 1 \\ 4 & 1 \end{vmatrix} = -3, \quad \begin{vmatrix} 0 & 2 \\ 1 & 3 \end{vmatrix} = -2 \tag{17}$$

Hence the rank of this matrix is 2

11.5 Simultaneous equations in matrix form

- Use matrix methods to solve simultaneous linear equations:

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \cdots & + & \cdots & + & \cdots & + & \cdots & = & \cdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array} \quad (18)$$

  where a_{ij} and b_i have known values, x_i are unknown.
- If the b_i are all zero, then the system of equations is called *homogeneous*, otherwise its *inhomogeneous*.
- We can write the set of equations as a matrix equation: $A\mathrm{x} = b$, (A is called the *coefficient matrix*). i.e.

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \cdots \\ b_m \end{pmatrix} \quad (19)$$

- We can define the *augmented matrix*

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{1n} & b_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right) \quad (20)$$

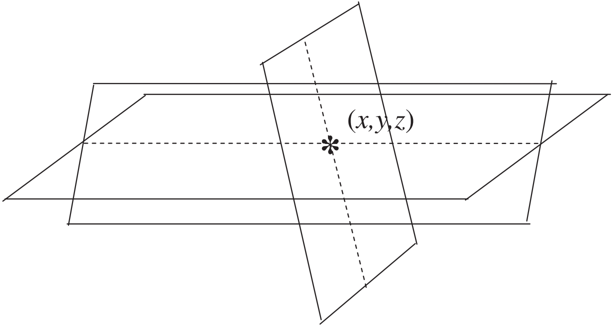
11.6 Unique solutions to simultaneous equations

- We have the case of n simultaneous equations in n unknowns: condition for the solution to be unique:
  - [Rank of coefficient matrix] = [Rank of augmented matrix] = = [Number of unknowns]
  - OR alternatively for the coefficient matrix $|A| \neq 0$ and $\underline{\mathbf{b}} \neq 0$.
- Note that $|A| \neq 0$ and $\underline{\mathbf{b}} = \mathbf{0}$ gives the trivial solution $(x_1, x_2, \cdots, x_n) = (0, 0, \cdots, 0)$.
- From here consider $n = 3$

$$\begin{array}{ccccccc} a_{11}x & + & a_{12}y & + & a_{13}z & = & b_1 \\ a_{21}x & + & a_{22}y & + & a_{23}z & = & b_2 \\ a_{31}x & + & a_{32}y & + & a_{33}z & = & b_3 \end{array} \qquad (21)$$

11.6.1 Geometrical interpretation

- The geometrical representation is 3 planes intersecting at a single point giving a unique solution.



- i.e. the scalar triple product $(a_{11}, a_{12}, a_{13}) \cdot [(a_{21}, a_{22}, a_{23}) \times (a_{31}, a_{32}, a_{33})] \neq 0$
- For the trivial solution $|A| \neq 0$ and $\underline{\mathbf{b}} = \mathbf{0}$, planes intersect at $(0, 0, 0)$