Classical Mechanics

LECTURE 10: RESISTED MOTION

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OUTLINE : 10. RESISTED MOTION

10.1 Resisted motion and limiting speed

- Newton II: $m\frac{dv}{dt} = F_{ext} + F_R$ where F_{ext} is the external force and F_R is a resistive force

- If $F_{ext} = 0$ and $F_R \propto$ velocity, then $v \propto \exp(-\alpha t)$ (see Lecture 4)

- If $F_{ext} \neq 0$ and e.g. $F_R \propto -v^n$ then there exists a *limiting speed* corresponding to $\frac{dv}{dt} = 0$ that satisfies $F_R = -F_{ext}$

10.2 Air resistance

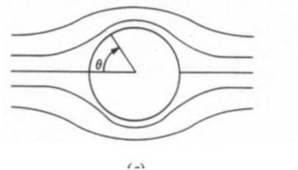
$$F_R = \underbrace{av}_{\text{Laminar flow}} + \underbrace{bv^2}_{\text{Turbulent flow}}$$

- Laminer flow : Stoke's Law
  $F = 6\pi\eta r v$

  r is the radius of the sphere

  v is the velocity of the sphere

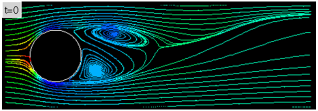
  η is the viscosity of the fluid



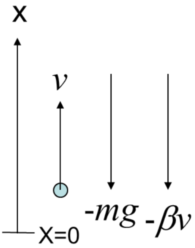
- Turbulent flow : S $F = \frac{1}{2}\pi\rho C_d r^2 v^2$

  ρ is the density of the fluid

  C_d is the drag coefficient (e.g. for a smooth sphere $C_d \sim 0.47$



10.3 Example 1 : Resistive force, $F_R \propto v$

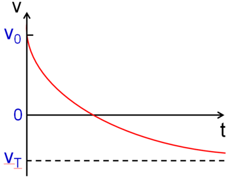


- Body fired *vertically upwards* under gravity $\rightarrow$ air resistance $\propto$ velocity

  $\rightarrow \; v = v_0$ & $x = 0$ at $t = 0$

- Equation of motion: $m\frac{dv}{dt} = -mg - \beta v$
  - $\int_{v_0}^{v} \frac{dv}{g+\alpha v} = -\int_0^t dt \quad$ where $\alpha = \frac{\beta}{m}$
  - $\left[\frac{1}{\alpha} log_e(g + \alpha v)\right]_{v_0}^{v} = [-t\,]_0^t$
  - $log_e\left(\frac{(g+\alpha v)}{(g+\alpha v_0)}\right) = -\alpha t \;\rightarrow\; 1+\frac{\alpha v}{g} = \left(1 + \frac{\alpha v_0}{g}\right)\exp(-\alpha t)$

$$v = \frac{g}{\alpha}\left[\left(1 + \frac{\alpha v_0}{g}\right)\exp(-\alpha t) - 1\right]$$



- Terminal (limiting) velocity: $t \rightarrow \infty$, $v_T \rightarrow -\frac{g}{\alpha}$
- Can show by expansion, as $\alpha \rightarrow 0$, $v \rightarrow v_0 - gt$

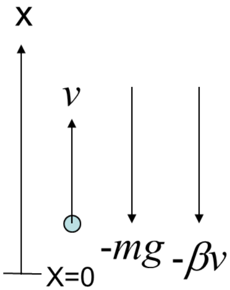
Maximum height and distance travelled for $F_R \propto v$

- $v = \frac{g}{\alpha}\left[(1+\frac{\alpha v_0}{g})\exp(-\alpha t) - 1\right]$
- At maximum height $\rightarrow v = 0, t = t_{max}$

  $\rightarrow \exp(-\alpha t_{max}) = (1+\frac{\alpha v_0}{g})^{-1}$

  $$t_{max} = \frac{1}{\alpha}\log_e\left(1+\frac{\alpha v_0}{g}\right)$$

  Can expand log to show :
  $t_{max} \rightarrow \frac{v_0}{g}$ when $\alpha \rightarrow 0$



- Distance travelled :
- $x = \int_0^t \frac{g}{\alpha}\left[(1+\frac{\alpha v_0}{g})\exp(-\alpha t) - 1\right] dt$

  $= \frac{g}{\alpha}\left[-\frac{1}{\alpha}(1+\frac{\alpha v_0}{g})\exp(-\alpha t) - t\right]_0^t$

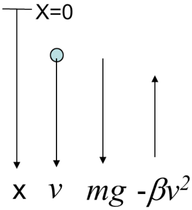
$$x = \frac{g}{\alpha}\left[\left(\frac{1}{\alpha}(1+\frac{\alpha v_0}{g})(1-\exp(-\alpha t)\right) - t\right]$$

Can show by expansion
$x \rightarrow v_0 t - \frac{1}{2}gt^2$
when $\alpha \rightarrow 0$

10.4 Example 2: Resistive force, $F_R \propto v^2$

- Body falls *vertically downwards* under gravity with air resistance $\propto$ [velocity]^2, $v = 0$, $x = 0$ at $t = 0$
- Equation of motion: $m\frac{dv}{dt} = mg - \beta v^2$
- Terminal velocity when $\frac{dv}{dt} = 0$: $v_T = \sqrt{\frac{mg}{\beta}}$
  - Equation of motion becomes $\frac{dv}{dt} = g\left(1 - v^2/v_T^2\right)$
  - Integrate $\int_0^v \frac{dv}{g\left(1-v^2/v_T^2\right)} = \int_0^t dt$
  - Standard integral : $\int \frac{1}{1-z^2}\, dz = \frac{1}{2}\log_e\left(\frac{1+z}{1-z}\right)$
  - $\left[\frac{v_T}{2g} log_e\left(\frac{1+v/v_T}{1-v/v_T}\right)\right]_0^v = t \;\rightarrow\; \frac{1+v/v_T}{1-v/v_T} = \exp(t/\tau)$, where $\tau = \frac{v_T}{2g}$

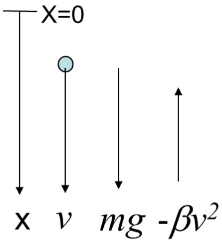
    $\rightarrow \; (1 - \frac{v}{v_T}) = (1 + \frac{v}{v_T})\exp(-\frac{t}{\tau})$



Velocity as a function of time:

$$v = v_T\left[\frac{1-\exp(-t/\tau)}{1+\exp(-t/\tau)}\right]$$

Velocity as a function of distance for $F_R \propto v^2$



- Equation of motion: $\frac{dv}{dt} = g\left(1 - v^2/v_T^2\right)$
- Write $\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$
  - $\int_0^v \frac{v\,dv}{g\left(1-v^2/v_T^2\right)} = \int_0^x dx$
  - $\left[-\frac{v_T^2}{2g} log_e\left(1 - v^2/v_T^2\right)\right]_0^v = x$

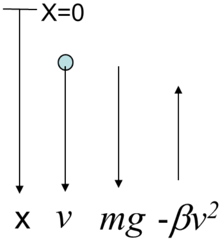
$\rightarrow \quad \left(1 - v^2/v_T^2\right) = \exp\left(-x/x_T\right) \quad , \quad \text{where } x_T = \frac{v_T^2}{2g}$

$$v^2 = v_T^2\left[1 - \exp\left(-x/x_T\right)\right]$$

To get x vs. t integrate again : $\rightarrow \int_0^t dt = \int_0^x \frac{dx}{v}$

10.4.1 Work done on the body by the force for $F_R \propto v^2$

- Equation of motion: $m\frac{dv}{dt} = mg - \beta v^2$
- Work done:
  $$\int F dx = \underbrace{\int_o^x mg\, dx}_{Conservative} - \underbrace{\int_0^x \beta v^2 dx}_{Dissipative}$$



- Conservative term : Work done $= mgx$
- Dissipative term : Work done
  $$= -\int_0^x \beta v^2 dx = -\int_0^x \beta v_T^2 \left[1 - \exp(-x/x_T)\right] dx$$
  $$= -\beta v_T^2 [x + \underbrace{x_T \left(\exp(-x/x_T) - 1\right)}_{= -x_T v^2/v_T^2}]$$
  $$= -\beta v_T^2 (x - v^2/2g) = -mg[x - v^2/2g]$$

$$v_T^2 = \frac{mg}{\beta}$$
$$x_T = \frac{v_T^2}{2g}$$

Energy dissipated = $\frac{1}{2}mv^2 - mgx$ As expected.