LECTURE 10: DETERMINANTS

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Outline: 10. DETERMINANTS

10.1 DETERMINANTS: What is a determinant?

- A determinant is a scalar, $det(A) \equiv |A|$, which is associated with any *square* matrix A (must be square).
- Start by simply *quoting* the determinants up to $n = 3$:
  - for a 1×1 matrix $\Rightarrow \quad det(A) = A$
  - for a 2×2 matrix

$$det(A) = det \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \equiv \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1 \quad (1)$$

  - for a 3×3 matrix

$$det(A) = det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \equiv \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \quad (2)$$

$$= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \quad (3)$$

$$= a_1 \times (b_2 c_3 - b_3 c_2) - a_2 \times (b_1 c_3 - b_3 c_1) + a_3 \times (b_1 c_2 - b_2 c_1)$$

10.2 Evaluating a general $N \times N$ determinant

- For an N $\times$ N matrix A, for each element A_{ij} we define a *minor* M_{ij}
- M_{ij} is the determinant of the (N-1) $\times$ (N-1) matrix obtained from A by deleting row i and column j.
- We also define *cofactor* $C_{ij} = (-1)^{(i+j)} M_{ij}$ (the "signed" minor of the same element).
- The determinant is then defined as the sum of the products of the elements of any row or column with their corresponding cofactors.

i.e. $det(A) = \sum_{j=1}^{N} A_{mj} C_{mj} = \sum_{i=1}^{N} A_{ik} C_{ik}$

for *ANY* row m or column k.

10.3 Evaluating Det of a 3×3 matrix "rigorously"

- Take the 3×3 matrix

$$|A| = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \quad (4)$$

- First knock out the first row and first column. Then the cofactor (1,1) is:

$$C_{11} = (-1)^{(1+1)} \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} \quad (5)$$

- Now get the next cofactor. Knock out the first row and second column:

$$C_{12} = (-1)^{(1+2)} \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} \quad (6)$$

- Then knock out the first row and third column:

$$C_{13} = (-1)^{(1+3)} \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \quad (7)$$

- This gives $det(A) = a_1 C_{11} + a_2 C_{12} + a_3 C_{13}$

More on the Det of a 3×3

- From before: $det(A) = a_1 C_{11} + a_2 C_{12} + a_3 C_{13}$
- Also, we can trivially show that the determinant is independent of the row or column chosen:

  e.g. via the 2^{nd} row: $det(A) = b_1 C_{21} + b_2 C_{22} + b_3 C_{23}$

  or via the 3^{rd} column: $det(A) = a_3 C_{13} + b_3 C_{23} + c_3 C_{33}$
- These equations are called *Laplace expansions* (or *Laplace developments*).

10.4 Extend to 4×4

$$|A| = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} \tag{8}$$

- Sum over 4 terms of cofactors: $|A| = \sum_{i=1}^{4} a_i C_{1i}$

$$= \quad a_1 \times (-1)^{(1+1)} \times \begin{vmatrix} b_2 & b_3 & b_4 \\ c_2 & c_3 & c_4 \\ d_2 & d_3 & d_4 \end{vmatrix} + a_2 \times (-1)^{(1+2)} \times \begin{vmatrix} b_1 & b_3 & b_4 \\ c_1 & c_3 & c_4 \\ d_1 & d_3 & d_4 \end{vmatrix} +$$

$$+ a_3 \times (-1)^{(1+3)} \times \begin{vmatrix} b_1 & b_2 & b_4 \\ c_1 & c_2 & c_4 \\ d_1 & d_2 & d_4 \end{vmatrix} + a_4 \times (-1)^{(1+4)} \times \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{vmatrix}$$

- Evaluating this will be very tedious if not done more efficiently. We strive to reduce elements of columns and/or rows to zero (see later).

10.5 Properties of determinants [1]

1. If we interchange 2 *adjacent* rows or 2 *adjacent* columns of A to give B, then $det(B) = -det(A)$

$$\text{e.g.} \quad \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = - \begin{vmatrix} a_2 & a_1 & a_3 \\ b_2 & b_1 & b_3 \\ c_2 & c_1 & c_3 \end{vmatrix} \qquad (9)$$

2. For a matrix A, the transpose satisfies

$$det(A^T) = det(A)$$

3. For a product of matrices then

$$det(AB) = det(BA) = det(A) \times det(B)$$

Properties of determinants [2]

4. If B results from multiplying one row or column of A by a scalar λ then $det(B) = \lambda \times det(A)$

$$\text{e.g.} \quad \lambda \times \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & \lambda a_2 & a_3 \\ b_1 & \lambda b_2 & b_3 \\ c_1 & \lambda c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ \lambda c_1 & \lambda c_2 & \lambda c_3 \end{vmatrix} \quad \text{etc.} \tag{10}$$

5. If I_n is an n $\times$ n unit matrix and λ a scalar, then

$$det(I_n) = 1 \quad \text{and} \quad det(\lambda I_n) = \lambda^n$$

Hence: $det(\lambda A) = det(\lambda I_n A) = det(\lambda I_n)\ det(A) = \lambda^n det(A)$

Properties of determinants [3]

6. For a matrix A, the inverse satisfies

$$det(A^{-1}) = 1/det(A)$$

Simple proof:

$$1 = det(I) = det(AA^{-1}) = det(A)\, det(A^{-1})$$
$$\rightarrow \quad det(A^{-1}) = 1/det(A) \qquad \text{QED}$$

7. For a matrix A where two or more rows (or columns) are equal or linearly dependent, then $det(A) = 0$

$$\text{e.g.} \quad \begin{vmatrix} a_1 & a_2 & a_3 \\ \lambda \times a_1 & \lambda \times a_2 & \lambda \times a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = 0 \quad (\text{where } \lambda \text{ is a scalar.}) \tag{11}$$

Properties of determinants [4]

8. If B results from adding a multiple of one row to another row, or a multiple of one column to another column, then $det(B) = det(A)$ (determinant unchanged).

$$\text{e.g.} \quad \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 + \lambda \times a_1 & a_3 \\ b_1 & b_2 + \lambda \times b_1 & b_3 \\ c_1 & c_2 + \lambda \times c_1 & c_3 \end{vmatrix} \quad \text{(where } \lambda \text{ is a scalar.)} \tag{12}$$

10.6 Evaluate a 4×4 determinant

$$\text{Evaluate} \begin{vmatrix} 1 & 3 & 0 & 4 \\ 3 & 7 & 2 & 9 \\ 1 & 0 & 2 & 4 \\ 0 & -1 & 4 & 3 \end{vmatrix} \tag{13}$$

Row 2 - (3 × row 1), put answer in row 2

$$= \begin{vmatrix} 1 & 3 & 0 & 4 \\ 0 & -2 & 2 & -3 \\ 1 & 0 & 2 & 4 \\ 0 & -1 & 4 & 3 \end{vmatrix} \tag{14}$$

Row 3 - row 1, put answer in row 3

$$= \begin{vmatrix} 1 & 3 & 0 & 4 \\ 0 & -2 & 2 & -3 \\ 0 & -3 & 2 & 0 \\ 0 & -1 & 4 & 3 \end{vmatrix} \tag{15}$$

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$$= \begin{vmatrix} 1 & 3 & 0 & 4 \\ 0 & -2 & 2 & -3 \\ 0 & -3 & 2 & 0 \\ 0 & -1 & 4 & 3 \end{vmatrix} \quad (16)$$

Laplace development of column 1
$$= (-1)^{(1+1)} \times 1 \times \begin{vmatrix} -2 & 2 & -3 \\ -3 & 2 & 0 \\ -1 & 4 & 3 \end{vmatrix} \quad (17)$$

Row 3 + row 1, put answer in row 3
$$= \begin{vmatrix} -2 & 2 & -3 \\ -3 & 2 & 0 \\ -3 & 6 & 0 \end{vmatrix} \quad (18)$$

Laplace development of column 3
$$= -3 \times (-1)^{(1+3)} \times \begin{vmatrix} -3 & 2 \\ -3 & 6 \end{vmatrix} \quad (19)$$

This is easily evaluated:
$-3 \times ((-3 \times 6) - (2 \times -3)) = 36$

10.7 The adjugate matrix

- The *adjugate matrix* is found by replacing each element of matrix A by its cofactor and then transposing the matrix. $Adj(A)_{ij} = C^T_{ij} = C_{ji}$

- Note: the adjugate is sometimes called the *adjoint*, but that terminology is rather ambiguous. "Adjoint" of a matrix normally refers to the Hermitian conjugate ($A^\dagger$).