LECTURE 1: INTRODUCING VECTORS

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OUTLINE : 1. INTRODUCING VECTORS

1.1 Scalars

1.2 Vectors

1.3 Unit vectors

1.4 Vector algebra

1.5 Simple examples

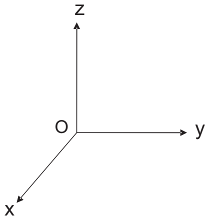
1.1 Scalars

- A scalar is a quantity with magnitude but no direction, any mathematical entity that can be represented by a number.
  - Examples: Mass, temperature, energy, charge ...
- Scalar addition, subtraction, division, multiplication are defined by the algebra of the real numbers representing the scalars.
  - $M_{total} = M_1 + M_2$, $V = I \times R$ etc.

But scalars are not enough if a physical observable has magnitude and direction.

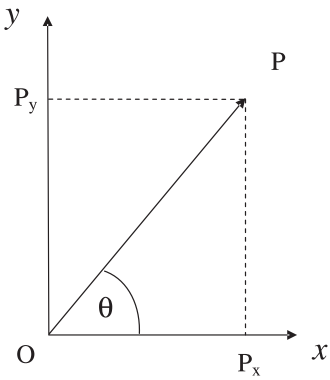
1.2 Vectors

- A *vector* is a quantity that has both magnitude (i.e. length) and direction. An "arrow" in space.
  - Examples: velocity, force, momentum, electric field etc.
- Need a reference frame (coordinate system). Use the Cartesian coordinate system defined by three orthogonal axes (in 3D).
- This is a right-handed rectangular coordinate system: A right threaded screw rotated clockwise through 90° from OX to OY will advance in the positive z direction.
- Vector then has magnitude $|\underline{\mathbf{p}}|$ and components (p_x, p_y, p_z)
- Convention **p** or $\underline{\mathbf{p}}$ or $\vec{p}$



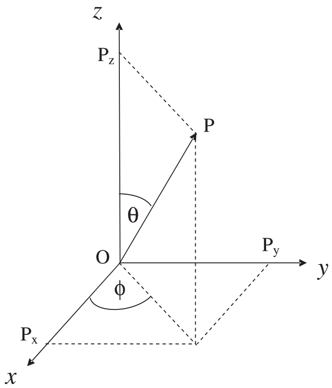
Vector components in 2D

- Projecting the components: $\underline{\mathbf{p}} = (p_x, p_y)$
- x-component $p_x = |\underline{\mathbf{p}}| \cos(\theta)$
- y-component $p_y = |\underline{\mathbf{p}}| \sin(\theta)$
- Magnitude $|\underline{\mathbf{p}}| = \sqrt{}(p_x^2 + p_y^2)$ measures the length of a vector.
- Direction $\tan(\theta) = (p_y / p_x)$



Vector components in 3D

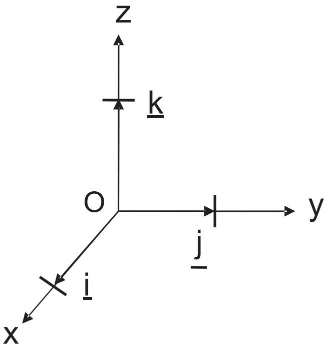
- Projecting the components:
  $\underline{\mathbf{p}} = (p_x, p_y, p_z)$
- x-component
  $p_x = |\underline{\mathbf{p}}| \sin(\theta) \cos(\phi)$
- y-component
  $p_y = |\underline{\mathbf{p}}| \sin(\theta) \sin(\phi)$
- z-component
  $p_z = |\underline{\mathbf{p}}| \cos(\theta)$
- Magnitude $|\underline{\mathbf{p}}| = \sqrt{(p_x^2 + p_y^2 + p_z^2)}$
- Direction $\tan(\phi) = (p_y/p_x)$
  $\cos(\theta) = (p_z/|\underline{\mathbf{p}}|)$



- Two vectors are equal if they have the same magnitude and direction
  $\underline{\mathbf{a}} = \underline{\mathbf{b}}$ gives $a_x = b_x, \; a_y = b_y, \; a_z = b_z$

1.3 Unit vectors

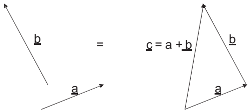
- A unit vector (sometimes called versor) is a vector with magnitude equal to one.
- e.g. Three unit vectors defined by orthogonal components of the Cartesian coordinate system:
  - $\underline{\mathbf{i}}$ = (1,0,0), obviously $|\underline{\mathbf{i}}| = 1$
  - $\underline{\mathbf{j}}$ = (0,1,0), $|\underline{\mathbf{j}}| = 1$
  - $\underline{\mathbf{k}}$ = (0,0,1), $|\underline{\mathbf{k}}| = 1$
- A unit vector in the direction of general vector $\underline{\mathbf{a}}$ is written $\underline{\hat{\mathbf{a}}} = \underline{\mathbf{a}}/|\underline{\mathbf{a}}|$
- $\underline{\mathbf{a}}$ is written in terms of unit vectors $\underline{\mathbf{a}} = a_x\underline{\mathbf{i}} + a_y\underline{\mathbf{j}} + a_z\underline{\mathbf{k}}$



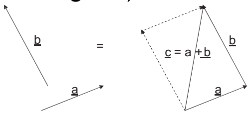
1.4 Vector algebra

Sum of two vectors

- To calculate the sum of two vectors
  $\underline{\mathbf{c}} = \underline{\mathbf{a}} + \underline{\mathbf{b}}$
  Triangle rule: Put the second vector nose to tail with the first and the resultant is the vector sum.
- $\underline{\mathbf{c}} = \underline{\mathbf{a}} + \underline{\mathbf{b}}$: in (x, y, z) components
  $(c_x, c_y, c_z) = (a_x + b_x, a_y + b_y, a_z + b_z)$
- Alternatively $\underline{\mathbf{c}} = \underline{\mathbf{a}} + \underline{\mathbf{b}}$
  $c_x\underline{\mathbf{i}} + c_y\underline{\mathbf{j}} + c_z\underline{\mathbf{k}} =$
  $(a_x + b_x)\underline{\mathbf{i}} + (a_y + b_y)\underline{\mathbf{j}} + (a_z + b_z)\underline{\mathbf{k}}$



(the parallelogram rule is completely analagous)



Multiplication of a vector by a scalar

- This gives a vector in the same direction as the original but of proportional magnitude.
- For any scalars α and β and vectors $\underline{\mathbf{a}}$ and $\underline{\mathbf{b}}$
  - $(\alpha\beta)\ \underline{\mathbf{a}} = \alpha(\beta\ \underline{\mathbf{a}}\ ) = \beta(\alpha\ \underline{\mathbf{a}}\ ) = \underline{\mathbf{a}}\ (\alpha\beta)$: commutative
  - $(\alpha+\beta)\underline{\mathbf{a}} = \alpha\underline{\mathbf{a}} + \beta\underline{\mathbf{a}}$: distributive
  - $\alpha(\underline{\mathbf{a}} + \underline{\mathbf{b}}) = \alpha\underline{\mathbf{a}} + \alpha\underline{\mathbf{b}}$

Vector algebra laws

- $\underline{\mathbf{a}} + \underline{\mathbf{b}} = \underline{\mathbf{b}} + \underline{\mathbf{a}}$: commutative law
- $\underline{\mathbf{a}} + (\underline{\mathbf{b}} + \underline{\mathbf{c}}) = (\underline{\mathbf{a}} + \underline{\mathbf{b}}) + \underline{\mathbf{c}}$: associative law
- Can treat vector equations in same way as ordinary algebra
  $\underline{\mathbf{a}} + \underline{\mathbf{b}} = \underline{\mathbf{c}} \quad \Rightarrow \quad \underline{\mathbf{a}} = \underline{\mathbf{c}} - \underline{\mathbf{b}}$
- Note that vector $-\underline{\mathbf{b}}$ is equal in magnitude to $\underline{\mathbf{b}}$ but in the opposite direction.
  so $\underline{\mathbf{a}} - \underline{\mathbf{b}} = \underline{\mathbf{a}} + (-\underline{\mathbf{b}})$

1.5 Simple examples

Example 1

Three forces $\underline{\mathbf{F}}_1$, $\underline{\mathbf{F}}_2$ and $\underline{\mathbf{F}}_3$ act on a particle so that the forces are in equilibrium. Given $\underline{\mathbf{F}}_1 = (-\underline{\mathbf{i}} + 2\underline{\mathbf{j}} + 4\underline{\mathbf{k}})$ N and $\underline{\mathbf{F}}_2 = (-\underline{\mathbf{i}} - 5\underline{\mathbf{k}})$ N find the third force.

- Forces must balance in all 3 directions $\underline{\mathbf{F}}_1 + \underline{\mathbf{F}}_2 + \underline{\mathbf{F}}_3 = 0$
- $\underline{\mathbf{F}}_3 = (+2\underline{\mathbf{i}} - 2\underline{\mathbf{j}} + \underline{\mathbf{k}})$ N
- $\underline{\mathbf{F}}_3$ has magnitude $\sqrt{}(2^2 + (-2)^2 + 1^2)) = \sqrt{}(9) = 3$ N

Example 2

Take a triangle. Point P divides AB in the ratio $\mu : \lambda$. What is the position vector of point P in terms of $\underline{\mathbf{a}}$ and $\underline{\mathbf{b}}$?

- $\underline{\mathbf{p}} = \underline{\mathbf{a}} + \frac{\lambda}{\mu+\lambda}\underline{\mathbf{AB}}$
- $\underline{\mathbf{b}} = \underline{\mathbf{a}} + \underline{\mathbf{AB}}$; $\underline{\mathbf{AB}} = \underline{\mathbf{b}} - \underline{\mathbf{a}}$
- $\underline{\mathbf{p}} = \underline{\mathbf{a}} + \frac{\lambda}{\mu+\lambda}(\underline{\mathbf{b}} - \underline{\mathbf{a}})$
- $\underline{\mathbf{p}} = (1 - \frac{\lambda}{\mu+\lambda})\underline{\mathbf{a}} + \frac{\lambda}{\mu+\lambda}\underline{\mathbf{b}}$
- $\underline{\mathbf{p}} = \frac{\mu}{\mu+\lambda}\underline{\mathbf{a}} + \frac{\lambda}{\mu+\lambda}\underline{\mathbf{b}}$