First-Year Electromagnetism: Problem Set 2

Hilary Term 2022, Prof Neville Harnew[1]

B. The Method of Image Charges

B.0 Background. Explain the factors that determine the parallel and normal components of the electric field near the surface of a flat metal plate.

B.1 Charge monopole near a flat metal surface. A point charge q is placed at a perpendicular distance d from a point O on a flat, infinite plate that is conducting and earthed.

(a) Use the method of images to show that the magnitude E of the electric field at the point P, a distance r from O in the plate, is given by

$$E(r) = \frac{qd}{2\pi\epsilon_0(r^2+d^2)^{3/2}}.$$

(b) Sketch the resulting lines of the electric field and calculate the force F between the charge and the plate.

(c) Show that the charge induced on the plate is $-q$.

(d) Find the work done in moving the charge to an infinite distance from the plate. Hence find the minimum energy an electron must have in order to escape from a metal surface (assume that it starts at a distance 0.1 nm, which is about one atomic diameter above the surface). Express your answer in electron volts.

B.2 Two charges near a flat metal surface. Two charges $+Q$ and $-Q$ are a horizontal distance a apart and a vertical distance b above a large grounded conducting sheet. Find the components of the forces acting on each charge.

B.3 Charge monopole near two orthogonal metal surfaces. Two semi-infinite grounded plane conducting plates are joined together at a right angle. A charge Q is situated near the join at a distance a from each plate.

(a) Show that the electric field is zero at any point along the join.

(b) Find the field just above the surface of one of the plates at the point closest to the charge.

(c) Sketch the equipotentials near the charge and near the plate.

(d) Calculate the surface charge density on the metal plates at the points closest to the charge Q.

(Hint: you need to consider three image charges.)

B.4 Uniformly charged rod near a metal surface. An infinite, thin, uniformly charged rod (line charge density λ) is placed parallel to a horizontal grounded metal plate a distance d above it. Point M lies on the plate vertically below the rod. Point P lies on the plate a distance x from point M, where line $\overline{PM}$ is perpendicular to the plane defined by the rod and point M. Use the result obtained in Problem A.4(b)[2] to calculate the electic field at P as a function of x.

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[1] With thanks to Prof Laura Herz

[2] i.e. $\underline{E} = \frac{\lambda}{2\pi\epsilon_0}\frac{\ell}{r\sqrt{\ell^2+r^2}}\,\hat{\underline{r}}$ (radially) with $\ell >> r$.

C. Electric Fields derived from Gauss' Law

C.0 Background. State Gauss' Law and explain how it may be used to determine the electric field arising from a spherically symmetric charge density distribution $\rho(r)$.

C.1 Uniformly charged sphere.

(a) Charge $+q$ is distributed uniformly throughout the volume of a sphere of radius a. Show that the electric field $\mathbf{E}$ and potential V at a distance r from the centre of the sphere are given by:

$$\mathbf{E} = \begin{cases} \dfrac{qr}{4\pi\epsilon_0 a^3}\,\hat{\mathbf{r}} \\[2ex] \dfrac{q}{4\pi\epsilon_0 r^2}\,\hat{\mathbf{r}} \end{cases} \quad \text{and} \quad V = \begin{cases} \dfrac{q}{4\pi\epsilon_0 a^3}\left(\dfrac{3a^2}{2} - \dfrac{r^2}{2}\right) & \text{for } 0 \le r \le a \\[2ex] \dfrac{q}{4\pi\epsilon_0 r} & \text{for } a \le r \end{cases}$$

(b) Repeat the calculation of fields and potentials for the charge now being uniformly distributed over the surface of a sphere of radius a.

(c) Draw graphs of the electric fields (magnitude) and the potentials for both cases (solid sphere and shell). Take care to illustrate the relation $E = -(\partial V/\partial r)$ everywhere and account for any discontinuities that occur at $r = a$.

C.2 Coulomb energy of the nucleus. The nucleus of an atom can be considered to be a charge $+Ze$ distributed uniformly throughout the volume of a sphere of radius a.

(a) Show that the potential energy W of a nucleus arising from the assembly of its uniform volume charge is given by $W = 3\,(Ze)^2(20\,\pi\epsilon_0 a)^{-1}$.

(b) What would the potential energy be if the charge was instead spread uniformly over the *surface* of the nucleus?

C.3 Electron in a hydrogen atom. From a quantum mechanical treatment, the potential at a distance r from the nucleus that is generated by an electron in a hydrogen atom is given by:

$$V = \frac{q}{4\pi\epsilon_0}\left(\frac{\exp(-2r/a) - 1}{r} + \frac{\exp(-2r/a)}{a}\right)$$

where a is a constant and is a measure of the "size" of the atom.

(a) Sketch $V(r)$ for $0 \le r \le \infty$ and comment on the shape of the curve.

(b) Find the magnitude of the electric field at a distance $r \ll a$ from the nucleus.

(c) Show that, when an external electric field E_{ext} is applied, the atom develops a dipole moment of magnitude p (you may assume that the electron cloud remains spherical and merely moves relative to the nucleus which has the same electric charge). By considering the force on the nucleus, calculate p and show that the polarisability p/E_{ext} is equal to $3\pi\epsilon_0 a^3$.

(d) For the hydrogen atom, $a = 0.5 \times 10^{-10}$ m. Show that even for the largest accessible fields of $\sim 10^6$ Vm^{-1} the electron charge cloud moves relative to the nucleus by only about 10^{-17} m (which justifies the assumption $r \ll a$).

(e) Use Gauss law to calculate the total charge in the cloud.