CP2 ELECTROMAGNETISM

https://users.physics.ox.ac.uk/~harnew/lectures/

LECTURE 9:

CAPACITANCE

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$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

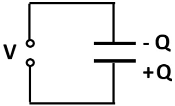
$$\frac{1}{\mu_0} \nabla \times \mathbf{B} = \mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

[1] With thanks to Prof Laura Herz

OUTLINE : 9. CAPACITANCE

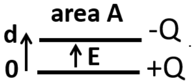
9.1 Capacitance

- Capacitors store electrostatic energy, by keeping two opposite charge accumulations on different metallic surfaces.



- Capacitance is defined as the charge that is stored per unit voltage applied between the two surfaces.

  Capacitance definition $C = \frac{Stored\ charge\ Q}{Voltage\ applied}$



- The charge is equal and opposite on both surfaces.
  - Simple example : Parallel plate capacitor
  - From before, $\underline{\mathbf{E}}$ constant between plates (Gauss) :

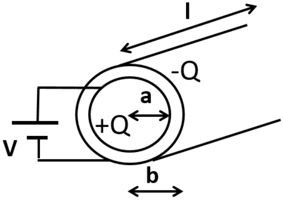
$$\oint_S \underline{\mathbf{E}} \cdot \underline{\mathbf{da}} = \frac{Q}{\epsilon_0} \rightarrow E = \frac{Q}{\epsilon_0 A}$$

$$V_{+-} = -\int_0^d E \cdot dx = -\frac{Q\,d}{\epsilon_0 A} \rightarrow V_{-+} = +\frac{Q\,d}{\epsilon_0 A}$$

$$C = \frac{Q}{V} = \frac{\epsilon_0 A}{d}$$

9.2 Cylindrical capacitor

- Example : coaxial cable. Battery supplies $+Q$ on the inner surface, $-Q$ is induced on the outer (Gauss)



From before, Gauss :

$\oint_S \underline{\mathbf{E}} \cdot \underline{\mathbf{da}} = E \cdot 2\pi r\, \ell = \frac{Q}{\epsilon_0}$

$\rightarrow\ \underline{\mathbf{E}} = \frac{Q/\ell}{2\pi\epsilon_0 r}\,\underline{\hat{\mathbf{r}}}$ (radial) for $a \le r \le b$

$\rightarrow\ E = 0$ for $0 \le r < a$ and for $r > b$

- $V_{+-} = -\int_a^b E_r \cdot dr = -\int_a^b \frac{Q/\ell}{2\pi\epsilon_0 r}\, dr$

$$= -\frac{Q/\ell}{2\pi\epsilon_0} log_e\left(\frac{b}{a}\right) \ \rightarrow\ V_{-+} = +\frac{Q/\ell}{2\pi\epsilon_0} log_e\left(\frac{b}{a}\right)$$

$$C = Q/V = \frac{2\pi\epsilon_0}{log_e(\frac{b}{a})} \times \ell$$

- Capacitance per unit length :

$$C' = C/\ell = \frac{2\pi\epsilon_0}{log_e(\frac{b}{a})}$$

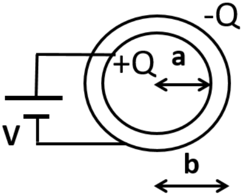
9.3 Spherical capacitor

- Example : spherical capacitor with concentric hollow spheres. Battery supplies $+Q$ on the inner sphere, $-Q$ is induced on the outer (Gauss).

From before, Gauss :

$\underline{\mathbf{E}} = \frac{Q}{4\pi\epsilon_0 r^2}\, \underline{\hat{\mathbf{r}}}$ (radial) for $a \leq r \leq b$

$E = 0$ for $r < a$ and $r > b$



- $V_{+-} = -\int_a^b E_r \cdot dr = -\int_a^b \frac{Q}{4\pi\epsilon_0 r^2}\, dr = \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{b} - \frac{1}{a}\right]$

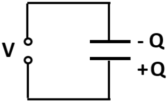
  $\rightarrow \quad V_{-+} = \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{a} - \frac{1}{b}\right]$

- Capacitance : $C = Q/V = 4\pi\epsilon_0 \frac{ab}{b-a}$

Capacitors summary

Capacitance: Storage of energy through separation of two oppositely poled charge accumulations

$$\text{Capacitance } C = \frac{\text{charge } Q}{\text{voltage } V \text{ applied}}$$



(i) parallel-plate capacitor:

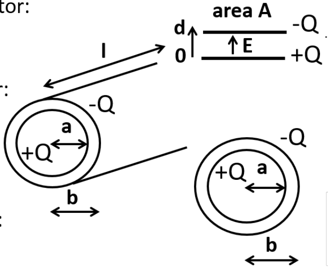
$$C = \varepsilon_0 \frac{A}{d}$$

(ii) cylindrical capacitor:

$$\frac{C}{l} = \frac{2\pi\varepsilon_0}{\ln(b/a)}$$

(iii) spherical capacitor:

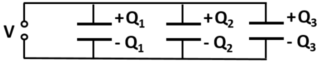
$$C = 4\pi\varepsilon_0 \frac{ab}{b-a}$$



9.4 Capacitance networks

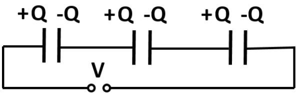
1. Capacitors in parallel

   - Voltage is the same across each capacitor.
   - Total charge :
     $$Q = Q_1 + Q_2 + Q_3 + \cdots$$
     $$= C_1 V + C_2 V + C_3 V + \cdots$$
   - Total capacitance
     $$C = \frac{Q}{V} = C_1 + C_2 + C_3 + \cdots$$



2. Capacitors in series

   - Charge is the same on each capacitor plate (inner plates are isolated from the outside world, with $Q_{tot} = 0$).
   - Total voltage :
     $$V = V_1 + V_2 + V_3 + \cdots$$
     $$\frac{1}{C} = \frac{V}{Q} = \frac{V_1}{Q} + \frac{V_2}{Q} + \frac{V_3}{Q} + \cdots$$
   - $$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \cdots$$



9.5 Energy stored in a capacitor

- Capacitor is initially uncharged : add a small amount of charge.
- Further charge will have to be brought up against the potential created by the existing charge :

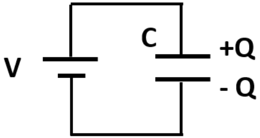
  Work done $\rightarrow \; dW = V(q)\, dq$
- Energy required to charge the capacitor to potential V_0 :

  $W = \int_0^{Q_0} V(q)\, dq$

  with $q = CV \;\rightarrow\; dq = C\, dV$
- $W = \int_0^{V_0} C\, V\, dV = \frac{1}{2}\, C\, V_0^2$
- Hence, energy stored in a capacitor with charge Q_0 and voltage V_0 :

$$U_C \equiv W = \frac{1}{2}\, C\, V_0^2 = \frac{1}{2}\, Q_0\, V_0 = \frac{1}{2}\, Q_0^2 / C$$

9.6 Changing C at constant V



- Battery maintains capacitor at constant V. What happens if C changes ?
- Energy stored in capacitor : $U_C = \frac{1}{2}\, C\, V^2$

  Change in capacitor energy : $dU_C = \frac{1}{2}\, V^2\, dC$
- Hence if C increases, U_C increases
- Since $Q = CV$, if C increases (ie. dC is positive), battery has to *supply* charge to maintain the same V. Hence charge on capacitor *increases*, and energy stored in battery *decreases*.
- Battery supplies dQ at constant $V \;\rightarrow$ energy change of battery is $dU_B = -V\, dQ$ (minus because battery *loses* stored energy in providing $+dQ$ to the plates of the capacitor)
- $Q = CV$, $dQ = V\, dC$, hence $dU_B = -V^2\, dC$
- This is a general result. If U_C *increases* at constant V, this is matched by a factor 2 *decrease* in battery energy.
- Cons. of energy : $dU_{total} = dU_B + dU_C = dW$, where

  $dW = -\frac{1}{2}\, V^2\, dC$ is the work done to change $C^{(*)}$.

(*) Note dC is negative if plates are pulled apart, since C decreases.