CP2 ELECTROMAGNETISM

https://users.physics.ox.ac.uk/~harnew/lectures/

LECTURE 8:

METHOD OF IMAGES

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$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\frac{1}{\mu_0} \nabla \times \mathbf{B} = \mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

[1] With thanks to Prof Laura Herz

OUTLINE : 8. METHOD OF IMAGES

8.1 The method of images

- The method of images is useful for calculating potentials created by charges placed in the vicinity of metal conductors.
- Replace conducting elements with imaginary charges (“image charges”) which replicate the boundary conditions of the problem on a surface.
- The Uniqueness Theorem guarantees that within the region bounded by the surface, the potential calculated for the “imagined” charge distribution is identical to that of the “real” situation.
- If a suitable replacement image charge distribution is chosen, the calculation of the potential becomes mathematically much simpler.

8.2 Example : Point charge above a metal plate

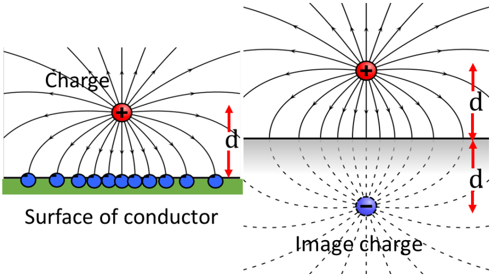
Point charge a distance d above a grounded metal plate:

- Boundary conditions

  1. At the metal surface (z = 0), the parallel component of $\underline{\mathbf{E}} = 0$. Field lines are perpendicular to the surface.

  2. Surface is an equipotential $\rightarrow \quad V = 0$

  3. Far from the charge and metal plate : $x^2 + y^2 + z^2 >> d^2 \quad V \rightarrow 0$



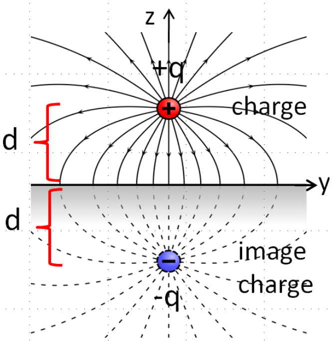
- The two configurations share the same charge distribution and boundary conditions for the upper volume half.
- The Uniqueness Theorem states that the potential in those regions must therefore be identical.
- In the upper half, the fields in both scenarios are identical.

Point charge above a metal plate, continued

- Above plate : real (physical) region.
  Here find solution at point (x, y, z).
- Below plate : imagined "mirror charge"
- $V(x,y,z) =$
  $$\frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{(x^2+y^2+(z-d)^2)}} - \frac{q}{\sqrt{(x^2+y^2+(z+d)^2)}}\right]$$
- Gives :
  1. $V = 0$ when $z = 0$
  2. $V \to 0$ for $x^2 + y^2 + z^2 >> d^2$

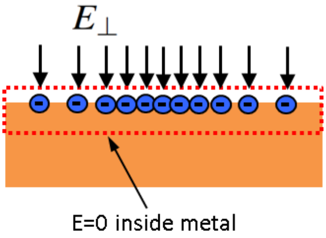
  $\to$ correct boundary conditions

  $\to$ unique solution!
- $\underline{\mathbf{E}}$ then calculated from $\underline{\mathbf{E}} = -\underline{\nabla} V$



8.3 Induced surface charge

- At the metal surface $E_{||} = 0$, $E_{\perp} = -\frac{\partial V}{\partial z}$ (z is the normal coordinate)
- Gauss Law at surface for element $\underline{\mathbf{da}}$: $\underline{\mathbf{E}} \cdot \underline{\mathbf{da}} = E_{\perp} \cdot da = \frac{q_{induced}}{\epsilon_0}$ where $\sigma_{induced} = q_{induced}/da$. No E-field in the "virtual" conductor



- So $\sigma_{induced}(x, y) = \epsilon_0 E_{\perp} = -\epsilon_0 \frac{\partial V}{\partial z}$
  - For the case of the point charge above the metal plate
  - $\frac{\partial V}{\partial z} = \frac{1}{4\pi\epsilon_0}\left[\frac{-2\times\frac{1}{2}\times q\,(z-d)}{(x^2+y^2+(z-d)^2)^{\frac{3}{2}}} - \frac{-2\times\frac{1}{2}\times q\,(z+d)}{(x^2+y^2+(z+d)^2)^{\frac{3}{2}}}\right]$
  - $\sigma_{induced}(x, y) = -\epsilon_0 \frac{\partial V}{\partial z}|_{z=0} = -\frac{1}{2\pi}\left[\frac{qd}{(x^2+y^2+d^2)^{\frac{3}{2}}}\right]$ this is the surface charge in the $x - y$ plane
  - $\sigma_{induced}$ is negative, and largest for $x = y = 0$

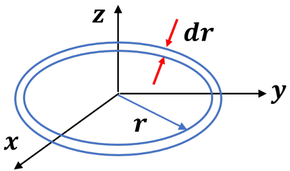
Total charge induced on the plate surface

- Now switch to polar coordinates (radial symmetry) :
- $\sigma_{induced} = -\frac{1}{2\pi}\left[\frac{qd}{(x^2+y^2+d^2)^{\frac{3}{2}}}\right] = -\frac{1}{2\pi}\left[\frac{qd}{(r^2+d^2)^{\frac{3}{2}}}\right]$

$$q_{induced} = \int_0^\infty \sigma(2\pi r\, dr)$$

$$= \int_0^\infty \left(-\frac{1}{2\pi}\right)\left[\frac{qd}{(r^2+d^2)^{\frac{3}{2}}}(2\pi r\, dr)\right]$$

$$= \frac{qd}{\sqrt{r^2+d^2}}\Big|_0^\infty = 0 - \frac{qd}{d} = -q$$



- The total charge induced on the plate is just $-q$, as would be expected.

8.4 Force between the charge and the plate & energy stored

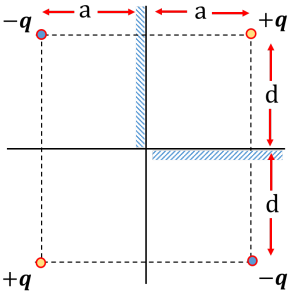
1. Force between the point charge and the plate :
   - Reduces to the case of the force between 2 point charges : $\underline{\mathbf{F}} = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{(2d)^2}\underline{\hat{\mathbf{z}}}$
2. Energy stored in the electric field
   - Bringing in charge from infinity - but noting the separation must always be maintained at $2z$ to preserve the geometry and potential of the plane. $W = -\int_{\infty}^{d}\underline{\mathbf{F}}\cdot\underline{\mathbf{d}\ell} = +\frac{1}{4\pi\epsilon_0}\int_{\infty}^{d}\frac{q^2}{(2z)^2}\,dz$
   - $W = -\frac{1}{4\pi\epsilon_0}\left(\frac{q^2}{(4z)}\right)\Big|_{\infty}^{d} = -\frac{1}{16\pi\epsilon_0}\frac{q^2}{d}$

   Compare this to the case of bringing two point charges together from infinity to separation $2d$, with the first charge fixed in space.

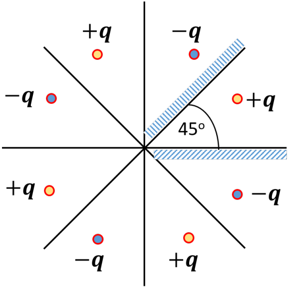
   - Bring charge $+q$ up to d needs no work $\left(V_{+} = \frac{+q}{4\pi\epsilon_0 r}\right)$
   - Second charge at $r = 2d$: $W = -q\,V_{+} = -\frac{q^2}{8\pi\epsilon_0 d}$
   - This is a factor 2 greater than bringing charge up to a plate

8.5 Image charges due to a pair of plates

1. Image charges required to replicate the field due a charge q located between 2 grounded semi-infinite plates with a 90° angle



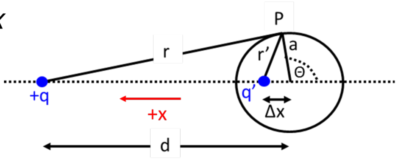
2. Image charges required to replicate the field due a charge q located between 2 grounded semi-infinite plates with a 45° angle



8.6 Point charge with grounded metal sphere

Look at a more complicated configuration $\cdots$ replicating the field due to a point charge outside a grounded metal sphere, radius a. Origin (0, 0, 0) is at sphere centre.

- Place image charge q' at position Δx from origin



- Potential at P on sphere :

$$V = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r} + \frac{q'}{r'}\right)$$

where $r^2 = a^2 + d^2 + 2\,a\,d\,\cos\theta$ &
$r'^2 = a^2 + \Delta x^2 + 2\,a\,\Delta x\,\cos\theta$

  - We need $V = 0$ for all points on the surface of the sphere (for all θ)

$$\frac{q}{\sqrt{a^2+d^2+2\,a\,d\,\cos\theta}} = -\frac{q'}{\sqrt{a^2+\Delta x^2+2\,a\,\Delta x\,\cos\theta}}$$

Point charge with grounded metal sphere, continued

$$\frac{q}{\sqrt{a^2+d^2+2\,a\,d\,\cos\theta}} = -\frac{q'}{\sqrt{a^2+\Delta x^2+2\,a\,\Delta x\,\cos\theta}}$$

- This can be solved rigorously by inputting specific values for $\cos\theta$ (eg, -1, 0, 1). However note that :
- LHS = $\frac{q}{\sqrt{a^2+d^2+2\,a\,d\,\cos\theta}} \times \underbrace{\left[\frac{a}{d}\Big/\sqrt{\frac{a^2}{d^2}}\right]}_{=1}$

$$= \frac{q\frac{a}{d}}{\sqrt{\frac{a^4}{d^2}+a^2+2\,\frac{a^3}{d}\,\cos\theta}}$$

- By inspection, image charge $q' = -q\frac{a}{d}$, at position $\Delta x = \frac{a^2}{d}$
- Hence potential at any point (x, y, z) OUTSIDE the sphere

$$V = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{(x-d)^2+y^2+z^2}} - \frac{q\,a/d}{\sqrt{(x-\frac{a^2}{d})^2+y^2+z^2}}\right]$$

(and $V = 0$ inside)