CP2 ELECTROMAGNETISM

https://users.physics.ox.ac.uk/~harnew/lectures/

LECTURE 8:

METHOD OF IMAGES

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$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\frac{1}{\mu_0} \nabla \times \mathbf{B} = \mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

[1] With thanks to Prof Laura Herz

OUTLINE : 8. METHOD OF IMAGES

8.1 The method of images

- The method of images is useful for calculating potentials created by charges placed in the vicinity of metal conductors.
- Replace conducting elements with imaginary charges ("image charges") which replicate the boundary conditions of the problem on a surface.
- The Uniqueness Theorem guarantees that within the region bounded by the surface, the potential calculated for the "imagined" charge distribution is identical to that of the "real" situation.
- If a suitable replacement image charge distribution is chosen, the calculation of the potential becomes mathematically much simpler.

8.2 Example : Point charge above a metal plate

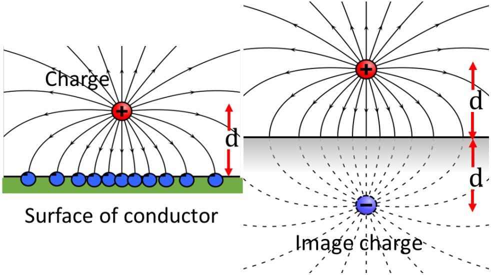
Point charge a distance d above a grounded metal plate:

- Boundary conditions

  1. At the metal surface (z = 0), the parallel component of $\underline{\mathbf{E}} = 0$. Field lines are perpendicular to the surface.

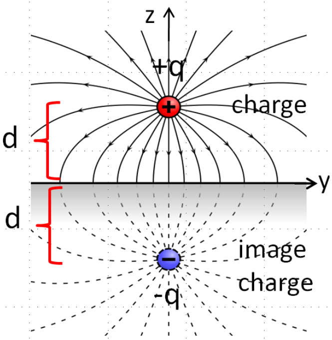
  2. Surface is an equipotential $\rightarrow \quad V = 0$

  3. Far from the charge and metal plate :
  $x^2 + y^2 + z^2 >> d^2 \quad V \rightarrow 0$

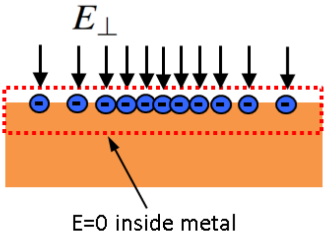


- The two configurations share the same charge distribution and boundary conditions for the upper volume half.
- The Uniqueness Theorem states that the potential in those regions must therefore be identical.
- In the upper half, the fields in both scenarios are identical.

Point charge above a metal plate, continued

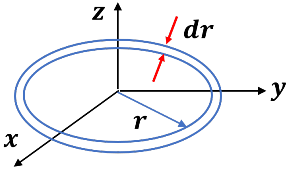


8.3 Induced surface charge



Total charge induced on the plate surface

- Now switch to polar coordinates (radial symmetry) :



- The total charge induced on the plate is just $-q$, as would be expected.

8.4 Force between the charge and the plate & energy stored

1. Force between the point charge and the plate :
   - Reduces to the case of the force between 2 point charges :

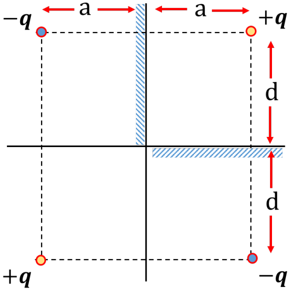
2. Energy stored in the electric field
   - Bringing in charge from infinity - but noting the separation must always be maintained at $2z$ to preserve the geometry of the plane.

Compare this to the case of bringing two point charges together from infinity to separation $2d$, with the first charge fixed in space.

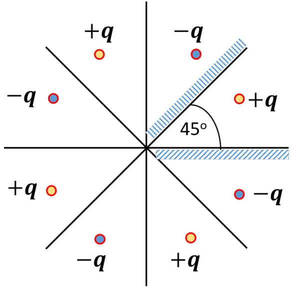
- Bring charge $+q$ up to d needs no work $\left(V_+ = \frac{+q}{4\pi\epsilon_0 r}\right)$
- Second charge at $r = 2d$: $W = -q\,V_+ = -\frac{q^2}{8\pi\epsilon_0 d}$
- This is a factor 2 greater than bringing charge up to a plate

8.5 Image charges due to a pair of plates

1. Image charges required to replicate the field due a charge q located between 2 grounded semi-infinite plates with a 90° angle



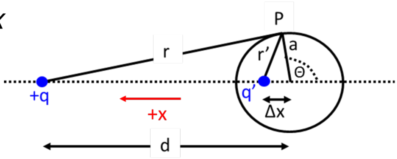
2. Image charges required to replicate the field due a charge q located between 2 grounded semi-infinite plates with a 45° angle



8.6 Point charge with grounded metal sphere

Look at a more complicated configuration $\cdots$ replicating the field due to a point charge outside a grounded metal sphere, radius a. Origin (0, 0, 0) is at sphere centre.

- Place image charge q' at position Δx from origin



Point charge with grounded metal sphere, continued

$$\frac{q}{\sqrt{a^2+d^2+2\,a\,d\,\cos\theta}} = -\frac{q'}{\sqrt{a^2+\Delta x^2+2\,a\,\Delta x\,\cos\theta}}$$

- This can be solved rigorously by inputting specific values for $\cos\theta$ (eg, -1, 0, 1). However note that :