CP2 ELECTROMAGNETISM

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LECTURE 7:

LAPLACE & POISSON EQUATIONS

Neville Harnew[1]

University of Oxford

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$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$
$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
$$\frac{1}{\mu_0}\nabla \times \mathbf{B} = \mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

[1] With thanks to Prof Laura Herz

OUTLINE :7. LAPLACE & POISSON EQUATIONS

7.1 Poisson and Laplace Equations

- The expression derived previously is the "integral form" of Gauss' Law

  $\oint_S \underline{\mathbf{E}} \cdot \underline{\mathbf{da}} = \frac{1}{\epsilon_0} \int_{\mathcal{V}} \rho \, d\mathcal{V}$ over volume $\mathcal{V}$

- We can express Gauss' Law in differential form using the Divergence Theorem :

  $\int_{\mathcal{V}} (\underline{\nabla} \cdot \underline{\mathbf{F}}) d\mathcal{V} = \oint_S \underline{\mathbf{F}} \cdot \underline{\mathbf{da}}$ [$\underline{\mathbf{F}}$ is any general vector field.]

  Hence $\int_{\mathcal{V}} (\underline{\nabla} \cdot \underline{\mathbf{E}}) d\mathcal{V} = \frac{1}{\epsilon_0} \int_{\mathcal{V}} \rho \, d\mathcal{V}$

- This gives $\underline{\nabla} \cdot \underline{\mathbf{E}} = \frac{\rho}{\epsilon_0}$

  the differential form of Gauss's Law

- Using $\underline{\mathbf{E}} = -\underline{\nabla} V$ get *Poisson's Equation* for potential V

  $\underline{\nabla}^2 V = -\frac{\rho}{\epsilon_0}$

- In regions where $\rho = 0$ we get *Laplace's Equation*:

  $\underline{\nabla}^2 V = 0$ (zero charge density)

7.2 Uniqueness Theorem

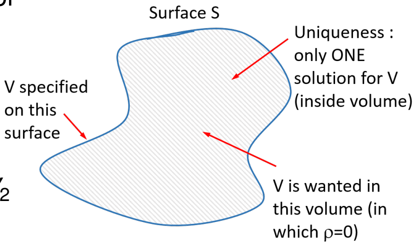
This states : *The solution to Laplace's equation in some volume is uniquely determined if the potential V is specified on the boundary surface S.* Why is this so?

- Suppose there are TWO solutions V_1 and V_2 to Laplace's equation for potential inside the volume
- $\underline{\nabla}^2 V_1 = 0$; $\underline{\nabla}^2 V_2 = 0$

  and $V_1 = V_2$ on the boundary surface S
- Define the difference $V_3 = V_1 - V_2$

  Then $\underline{\nabla}^2 V_3 = \underline{\nabla}^2 V_1 - \underline{\nabla}^2 V_2 = 0$

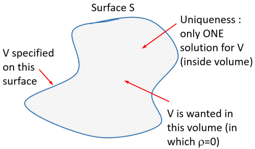
  (V_3 also obeys Laplace's equation)
- But on the boundary $V_3 = V_1 - V_2 = 0$



Uniqueness Theorem continued

From the previous page :

- $\underline{\nabla}^2 V_1 = 0$ & $\underline{\nabla}^2 V_2 = 0$ with $V_1 = V_2$ on the surface
- $V_3 = V_1 - V_2$ (which $= 0$ on the surface)
- $\underline{\nabla}^2 V_3 = 0$ everywhere.



  - The $\underline{\nabla}^2$ operator is a three-dimensional second derivative of a function - when a function has an extrema, the second derivative will be negative for a maximum and positive for a minimum.
  - The fact that the second derivative is always zero therefore indicates that there are no such minima or maxima in the region of interest
  - Hence solutions to Laplace's equation do not have minima or maxima.
  - Since $V_3 = 0$ on the surface, the maximum and minimum values of V_3 must also be zero everywhere inside it.

Hence $V_3 = 0$ everywhere, and **V must be unique**

  - Note the same applies to Poisson's equation.
  - If $\underline{\nabla}^2 V_1 = -\rho/\epsilon_0$ and $\underline{\nabla}^2 V_2 = -\rho/\epsilon_0$, then $\underline{\nabla}^2 V_3 = 0$ as before.

Poisson and Laplace Equations : summary

Gauss' law: $\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$ Definition of Potential: $\mathbf{E}(\mathbf{r}) = -\nabla V(\mathbf{r})$

$$\nabla^2 V = -\frac{\rho}{\varepsilon_0}$$ Poisson equation

In regions where ρ=0: $\nabla^2 V = 0$ Laplace equation

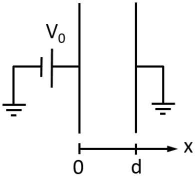
Uniqueness Theorem:
The potential V inside a volume is *uniquely* determined, if the following are specified:

(i) The charge density throughout the region

(ii) The value of V on all boundaries

7.3 Laplace equation in cartesian coordinates

Example : Solutions to Laplace's equation for a parallel plate capacitor. Symmetry suggests use of cartesian coordinates.



- $\frac{\partial^2 V}{\partial x^2} + \underbrace{\frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}}_{= 0 \text{ (by symmetry)}} = 0$

  Need to solve $\frac{\partial^2 V}{\partial x^2} = 0$
- $\frac{\partial V}{\partial x} = C_1 \rightarrow V(x) = C_1 x + C_2$
- Values on boundary defined by capacitor plates :

  $V(x=0) = V_0$ and $V(x=d) = 0$
- $x = 0$, $C_2 = V_0$ and

  $x = d$, $C_1 d + C_2 = 0 \rightarrow C_1 = -V_0/d$
- Solution : $V(x) = V_0(1 - x/d)$
- Electric field $\underline{\mathbf{E}} = -\nabla V = -\frac{\partial}{\partial x} V \underline{\hat{\mathbf{x}}} \rightarrow \underline{\mathbf{E}} = \frac{V_0}{d} \underline{\hat{\mathbf{x}}}$

7.4 Laplace Equation in spherical coordinates

... assuming azimuthal symmetry.

General solutions to Laplace's equation for charge distributions with azimuthal symmetry (mainly for information here : see second year).

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \underbrace{\frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}}_{=0} = 0$$

Separation of variables yields the general solutions:

$$V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$$

where A_l, B_l are constants determined by boundary conditions and P_l are Legendre Polynomials in cos θ, i.e.:

$P_0(\cos\theta) = 1$
$P_1(\cos\theta) = \cos\theta$
$P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1)$
Etc ...

$$V(r,\theta) = A_0 + \frac{B_0}{r} + A_1 r\cos\theta + \frac{B_1}{r^2}\cos\theta$$

$$+ A_2 r^2\frac{1}{2}(3\cos^2\theta - 1) + \frac{B_2}{r^3}\frac{1}{2}(3\cos^2\theta - 1) + \cdots$$

Laplace equation examples in spherical coordinates

1. Take a defined small spherical volume which contains some azimuthally symmetric charge distribution :
   - Outside the volume $\rho = 0$
   - Boundary condition on potential : $V \rightarrow 0$ as $r \rightarrow \infty$
   - Hence $A_\ell = 0$ for all ℓ
   - Retain just multipole expansion terms (monopole + dipole+ quadrupole + $\cdots$ terms)
2. Special case of spherically symmetric charge distribution inside the volume :
   - Outside the volume $\rho = 0$, $\nabla^2 V = 0$ with no θ dependence
   - $A_\ell = B_\ell = 0$ for $\ell \neq 0$
   - $V(r) = A_0 + B_0/r$ as expected from Gauss' Law